Unit - I

panchram

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Product space

Imp

Que/ Any product of ~~compact space~~ Hausdorff space is Hausdorff.

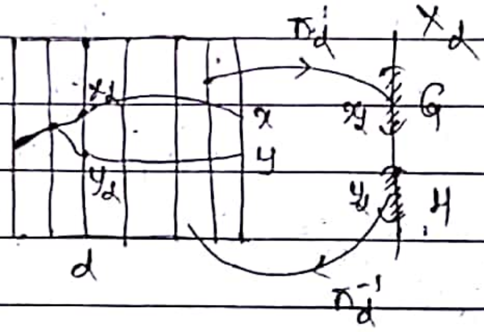
Proof - Suppose $(X_j, T_j)_{j\in I}$ is the arbitrary family of Hausdorff space. Take $X = \prod_{j\in I} X_j$ endowed with product topology.

Consider $x, y \in X$ such that $x \neq y$, where $x = (x_j)_{j\in I}$, $y = (y_j)_{j\in I}$

$\Rightarrow \exists\ \alpha \in I$ such that $x_\alpha \neq y_\alpha$, where $x_\alpha, y_\alpha \in X_\alpha$ and X_α is Hausdorff

$\Rightarrow \exists$ two open set G, H in X_α such that $x_\alpha \in G$, $y_\alpha \in H$ and $G \cap H = \phi$ in X_α.

Consider $U = \pi_\alpha^{-1}(G)$
$V = \pi_\alpha^{-1}(H)$



$\Rightarrow U$ and V are open set in X because each projection π_α is continuous such that

$x \in U$, $y \in V$ and $U \cap V = \pi_\alpha^{-1}(G) \cap \pi_\alpha^{-1}(H)$

$$= \left(\prod_{j\in I\setminus\{\alpha\}} X_j \times G\right) \cap \left(\prod_{j\in I\setminus\{\alpha\}} X_j \times H\right)$$

$$= \left(\prod_{j\in I\setminus\{\alpha\}} X_j\right) \times (G \cap H)$$

$$= \prod_{j\in I\setminus\{\alpha\}} X_j \times \phi = \phi$$

Hence X is ~~comp~~ Housdorff. proved.

Lemma - Any product of closed set is closed.

Proof - ~~Suppose~~ let $\{C_j\}_{j\in I}$ be a family of closed set where $C_j \in X_j$ $\forall$ $j \in I$.

Take $C = \prod_{j\in I} C_j$ and let $x \notin C$

Now $C = \bigcap_{j\in I=\{j_1, j_2, \ldots\}} \pi_j^{-1}(C_j)$

$\Rightarrow$ $\exists$ $j \in I$ such that $\pi_j(x) \notin \pi_j^{-1}(C_j)$, $C_j \subset X_j$

Set $V_j = X_j - C_j$, clearly V_j is an open set in X_j such that $\exists$ $\pi_j(x) \in V_j$

$\Rightarrow$ $\exists$ $C_j \cap V_j = \phi$ Jx

~~claim~~ ~~is~~ moreover $C \cap V = \phi$, $\boxed{\text{since } \pi_j(C) \cap \pi_j(V) = \phi}$

$\Rightarrow$ $x \in V \subset X - C$ $\quad [V = \pi_j^{-1}(V_j)]$

$\Rightarrow$ $x \in X - C$, Then ~~$\exists$ an open set V s.t.~~ $X - C$ is a nbd of each of its points.

$\Rightarrow$ $X - C$ is open

$\Rightarrow$ $C = \prod C_j$ is closed. proved.

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$\Rightarrow$ (continues)
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Show that each projection function is open.

Proof - Suppose $X = \prod_{j\in I} X_j$ is endowed with product topology.

$\Rightarrow$ $\pi_j : X \longrightarrow X_i$ is continuous.

Consider G as an open set in X containing x

$\Rightarrow$ $\exists$ a basis open set, say $V = \prod_{j\in I} V_j$, where $V_i = X_i$ for i except $i = i_1, i_2, \ldots i_n$

i.e. $V_{j_r} \subsetneq X_{j_r}$, $r = 1, 2, \ldots n$

Such that $x \in V \subset G$

$\Rightarrow$ $\pi_j(x) \in V_j \subset \pi_i(G)$.

$\Rightarrow$ $\pi_i(G)$ is a nbd of ~~it~~ ~~each of e~~ its typical point $\pi_i(x)$

$\Rightarrow$ The function π_i is open. proved.

If $\Pi(X_i : i \in I)$ the function $\pi_i : X \to X_i$ defined by $\pi_i(x) = x_i$ $\forall$ $x \in X$ is said to be projection function.

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Any product of regular space is regular.

Proof:- Let $X = \prod_{j \in I} X_j$ be the endowed with product topology.

Let C be a closed set such that $x \notin C$

$\Rightarrow$ $X - C$ is an open set containing x.

$\Rightarrow$ $\exists$ a basis open set say $V = \prod_{j \in I} V_j$, $V_{j_r} \subsetneq X_{j_r}$ $r = 1, 2, \dots n$

and $V_j = X_j$ $\forall$ $j \in I - \{j_1, j_2, \dots, j_n\}$ such that

$x \in V \subset X - C$

Then $x_{j_r} = \pi_{j_r}(x) \in V_{j_r}$ $(r = 1, 2, \dots n)$, X_{j_r} being regular space.

$\Rightarrow$ $\exists$ an open set U_{j_r} such that $x_{j_r} \in U_{j_r}$, $\overline{U}_{j_r} \subset V_{j_r}$

Set $U = \prod_{j \in I} U_j$, Then we can show that

$\overline{U} = \prod_{j \in I} \overline{U_j}$ $\longrightarrow$ (1)

and $x \in U \subset \overline{U}$, $\overline{U} \subset V$ $\longrightarrow$ (2)

Since $V \subset X - C$, $\overline{U} \subset V$

$\Rightarrow$ $C \subset X - V$, $X - V \subset X - \overline{U}$

$\Rightarrow$ $C \subset X - \overline{U}$ $\longrightarrow$ (3)

from (2) and (3),

Sierpinski space is not metrizable.

$\overline{U} \cap X - \overline{U} = \phi$

$\Rightarrow$ $U \cap X - \overline{U} = \phi$

So $x \in U$, $C \subset V$ $\{V = X - \overline{U}\}$

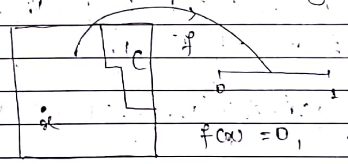
and $U \cap V = \phi$.

Hence product of regular space is regular.

proved.

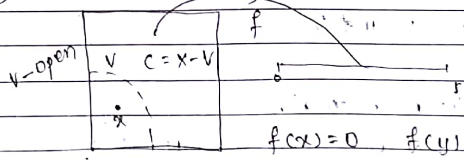
Completely regular - A space X is said to be completely regular if for any point $x \in X$ and closed set C not containing x, there exist a continuous function $f: X \to [0, 1]$ s.t. $f(x) = 0$, and $f(y) = 1$ $\forall$ $y \in C$.

(I) def^n



$f(x) = 0$, $f(y) = 1$, $\forall$ $y \in C$

def^n (II)



$f(x) = 0$, $f(y) = 1$, $\forall$ $y \notin V$.

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& X is completely regular iff for any subbasis set S, $V \in S$, $x \in V$, $\exists$ a continuous function $f : X \to [0,1]$ such that $f(x) = 0$, $f(y) = 1$ $\forall$ $y \notin V$.

Proof - If X is completely regular, then certainly the condition holds for all open sets and for some members of S.

Conversely, let G be an open set in X and let $x \in G$. By definition of a sub-base there exist $V_1, V_2, \dots, V_n \in S$ such that

$$x \in \bigcap_{i=1}^{n} V_i \subset G, \text{ for each } i = 1, 2, \dots, n$$

$$\Rightarrow x \in V_i, \; i = 1, 2, \dots, n$$

By hypothesis, $\exists$ a continuous function $f_i : X \to [0,1]$ such that $f_i(x) = 0$, $f_i(y) = 1$ $\forall$ $y \notin V_i$, $i = 1, 2, \dots n$

then we define $f : X \to [0,1]$ by

$$f(x) = 1 - (1 - f_1(x))(1 - f_2(x)) \dots (1 - f_n(x))$$

clearly, f is continuous and $f(x) = 0$

and $f(y) = 1$ $\forall$ $y \notin G$

$\Rightarrow X$ is completely regular.

proved.

Imp

Theorem - The product of completely regular spaces is completely regular, iff each coordinate space is so.

Proof ($\Rightarrow$) let $\pi_i : X \to X_i$ be continuous and complete regularity is a topological property. So that X_i is also completely regular ($X_i = \pi_i(x)$) $\to$ (1)

($\Leftarrow$) Suppose each X_i is completely regular, then we have to show that $X \; (= \prod_{i \in I} X_i)$ is completely regular.

let $S = \{ \pi_i^{-1}(V_i) \mid i \in I, V_i \text{ open in } X_i \}$ be the standard sub-base for the product topology.

let $x \in \pi_i^{-1}(V_i)$, then $\pi_i(x) \in V_i$ and so by complete regularity of X_i, $\exists$ $f_i : X_i \to [0,1]$ such that

$$f_i(\pi_i(x)) = 0, \; f_i(\pi_i(y)) = 1 \;\; \forall \; y \notin V_i \quad \to (2)$$

from eq^n (1) and (2), we define

$f_i \circ \pi_i : X \xrightarrow{\text{continuous}} [0,1]$, Then we see that

$$(f_i \circ \pi_i)(x) = f_i(\pi_i(x)) = 0$$

$$(f_i \circ \pi_i)(y) = f_i(\pi_i(y)) = 1 \quad \forall \; y \notin \pi_i^{-1}(V_i).$$

Hence X is completely regular.

proved.

A product of topological space is connected iff each component space is so.

OR

$(X = \prod_{i\in I} X_i, T)$ is connected $\Leftrightarrow$ each (X_i, T_i) is connected, $\forall i \in I$.

Proof $(\Rightarrow)$ Suppose $X = \prod_{i\in I} X_i$ is connected, take $\pi_i : X \to X_i$ is a projection function.

Since each projection function is continuous, so π_i is continuous. Moreover $X_i = \pi_i(X)$, i.e. component space X_i is a continuous image of a connected space X and so "every continuous image of connected space is connected."

Hence each X_i is connected.

$(\Leftarrow)$ Conversely - Suppose each X_i is connected, for $i \in I$. Then we have to show that X is connected.

Let $x \in X = \prod_{i\in I} X_i$ and let C be a component of X containing x; $x = \{x_i\}_{i\in I}$. Let G be an open set in X. Then $\exists$ a basis open set, say V such that,

$$V \subset G, \text{ where } V = \prod_{i\in I} V_i$$

and

$$V_i = X_i, \text{ for } i \in I, \text{ and } i \neq i_1, i_2, \ldots, i_n$$

$$V_i \subsetneq X_i \text{ for } i = i_1, i_2, \ldots, i_n$$

Let $Z = \prod_{i\in I} Z_i$, where $Z_i = \{x_i\}$ for $i \neq i_1, i_2, \ldots, i_n$

and $Z_i = X_i$, $i = i_1, i_2, \ldots, i_n$

where x_i's are arbitrary element in X_i ($i \neq i_1, i_2, \ldots, i_n$)

Observation -

(I) $x \in Z$, since $Z_i = \{x_i\}$ for $i \neq i_1, i_2, \ldots, i_n$

$\Rightarrow (x_i) \in \prod_{i\in I} Z_i$

$\Rightarrow x \in \prod_{i\in I} Z_i$

$\Rightarrow x \in Z$

(II) $Z \simeq \prod_{r=1}^{n} X_{i_r} = X_{i_1} \times X_{i_2} \times \cdots \times X_{i_n}$, since each X_i connected $\Rightarrow Z$ is connected, since finite cartesian product of connected space is connected.

So $x \in C$ and $x \in Z$ also

$\Rightarrow x \in C \cap Z$

Since C and Z both are connected and $C \cap Z \neq \phi$

$\Rightarrow Z \cup C$ is connected. $\{$since $\mathcal{C} = \{C_i\}_{i\in I}$ each connected$\}$ and $C_i \cap C_j \neq \phi$, $i, j \in I \Rightarrow \bigcup_{i\in I} C_i$ is connected.

But C is component, it is a maximal ~~element~~ connected subset of X, so

$$Z \cup C = C \Rightarrow Z \subseteq C$$

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(III) $Z \cap V \neq \phi$, let $y = (y_i)_{i \in I}$ where $y_i = x_i$ for $i \neq i_1, i_2, \ldots, i_n$ and y_i is any arbitrary element of V_i, for $i = i_1, i_2, \ldots, i_n$

But by definition of $V = \prod_{i \in I} V_i$, $V_i = X_i$ for $i \neq i_1, i_2, \ldots i_n$

$\Rightarrow$ $V_i \subsetneq X_i$ for $i = i_1, i_2, \ldots, i_n$

$\Rightarrow$ $y_i \in V_i$, for $i = i_1, i_2, \ldots i_n$

$\Rightarrow$ $y = (y_i) \in \prod_{i \in I} V_i$

$\Rightarrow$ $y \in V$

$\Rightarrow$ $Z \cap V \neq \phi$

further, we see that $V \subseteq G$

$\Rightarrow$ $Z \cap G \neq \phi$, but $Z \subseteq C$

$\Rightarrow$ $C \cap G \neq \phi$, but G was arbitrary open set

$\Rightarrow$ C is dense in X

$\Rightarrow$ $\overline{C} = X$

$\Rightarrow$ X is connected.

This completes the proof of the theorem.

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Theorem - Prove that the product space $X_1 \times X_2$ is connected iff X_1 and X_2 are connected.

Proof - ($\Rightarrow$) Suppose $X = \prod_{i \in I} X_i$ is connected $I = \{1, 2\}$

i.e. $X = X_1 \times X_2$

Take $\pi_1 : X \to X_1$ and $\pi_2 : X \to X_2$ where π_1 and π_2 are projection function and both are continuous and open. Moreover $X_1 = \pi_1(X)$ and $X_2 = \pi_2(X)$ i.e. component space X_1 and X_2 are continuous image of a connected space X and hence continuous image of connected space is connected.

Thus X_1 and X_2 are connected.

($\Leftarrow$) Conversely, suppose X_1 and X_2 are connected. let $x \in X = X_1 \times X_2$ and let C be a component of X containing x

Suppose T_1 and T_2 are two topology of X_1 and X_2 respectively, and let (X, T) is a product topology.

Let $(x_1, x_2) \in X$ and consider the sets $\{x_1\} \times X_2$ and $X_1 \times \{x_2\}$

we define $f : \{x_1\} \times X_2 \to X_2$ by

$f((x_1, x_2)) = x_2 \ \forall\ x_2 \in X_2$

and $g : X_1 \times \{x_2\} \to X_1$ ...

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Clear that f and g are homeomorphism. Consequently f^{-1} and g^{-1} are continuous maps so that $f^{-1}(X_2)$ and $g^{-1}(X_1)$ are connected sets

i.e. $X_{x_1} = \{x_1\} \times X_2$, $X_{x_2} = X_1 \times \{x_2\}$

and $C_{x_1 x_2} = X_{x_1} \cup X_{x_2}$

Then X_{x_1} & X_{x_2} are connected sets.

$(x_1, x_2) \in X_{x_1} \cup X_{x_2}$ so that $X_{x_1} \cap X_{x_2} \neq \phi$

Being a finite union of connected sets having non-empty intersection $X_{x_1} \cup X_{x_2}$ is connected. i.e. $C_{x_1 x_2}$ is connected.

Consider the family of connected sets

$\{ C_{x_1 x_2} : C_{x_1 x_2} \in X \}$ ⟶ (1)

$\cup \{ C_{x_1 x_2} : C_{x_1 x_2} \in X \} = \cup \{ X_{x_1} \cup X_{x_2} \} = X_1 \times X_2 = X$

Let $(a_1, a_2) \in X$ be fixed

Then $C_{a_1 a_2}$ is fixed member of the family (1).

$C_{a_1 a_2}$ meets $C_{x_1 x_2}$ in points (a_1, x_2) and (x_1, a_2)

⇒ $C_{a_1 a_2} \cap C_{x_1 x_2} \neq \phi \quad \forall (x_1, x_2) \in X$

⇒ $\cup C_{x_1 x_2} = X$

⇒ X is connected. proved

Imp

Theorem – Metrizability is a countably productive property.

OR

Any countable product of metric space is a metric space.

Proof – ~~Suppose~~ Let $\{(X_n, d_n) : n \in N\}$ be a countable collection of metric space s.t. $\mathcal{T}_n = \mathcal{T}_{d_n}$ be corresponding topologies of X_n induced by d_n itself. and let $(X, \mathcal{T})$ be the topological product of $\{(X_n, d_n) : n \in N\}$, We have to prove that there is a metric d on X which induces the topology $\mathcal{T}$.

Define bounded metric e_n on X_n satisfying the condition $e_n(x, y) \le 2^{-n} \quad \forall \ n \in N$

Denote points of $X = \prod_{n \in N} X_n$ by sequence $x = \{x_n\}$ with $x_n \in X_n$, $\forall \ n \in N$.

Now define $d : X \times X \to R$ by

$$d(x, y) = \sum_{n=1}^{\infty} d_n(x_n, y_n) \longrightarrow (1)$$

Claim – d is well defined. {i.e. This series is convergent}

Let either $d_n \le 2^{-n}$ or we can replaced ~~to~~ d_n by e_n which bounded 2^{-n}

i.e. $d_n(x_n, y_n) \le 2^{-n} \quad \forall \ n \in N$

⇒ $\sum_{n=1}^{\infty} d_n(x_n, y_n) \le \sum_{n=1}^{\infty} 2^{-n} \le \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots \infty\right)$

$\Rightarrow \sum_{n=1}^{\infty} d_n(x_n, y_n) \le 1$

Hence $d(x,y) < 1 \Rightarrow d(x,y) \in \mathbb{R} \Rightarrow d$ is well-defined $\mathcal{T}_d$ on X

<u>claim</u> $\mathcal{T}_d = \mathcal{T}\tau$. i.e. (X, τ) is metrizable.

For this.

let G be an open set in metric space (X, d) then $G \in \mathcal{T}_d$ and let $x \in G \in \mathcal{T}_d$, then $\exists\ r > 0$ s.t. $B(x, r) \subset G$.

Now we intend to show that G is τ-open. It suffices to show that G is a nbd of each of its point.

i.e. let V be open basis $x \in V \subset G$ or:

We choose N so large that $\sum_{n=N+1}^{\infty} 2^{-n} < r/2$

$$* \left[\sum_{n=N+1}^{\infty} 2^{-n} = \frac{1}{2^{N+1}} + \frac{1}{2^{N+2}} + \dots = \frac{1/2^{N+1}}{1 - 1/2^{N+1}} \longrightarrow 0 \text{ as } N \to \infty \right]$$

So $\sum_{n=N+1}^{\infty} 2^{-n} < r/2 \longrightarrow (2)$

We constructs an open balls $V_n = B_n\left(x_n, \frac{r}{2N}\right)$, $n = 1, 2, \dots N$ $\longrightarrow (3)$

set $V = \prod_{n \in \mathbb{N}} V_n$, $V_n = X_n \ \forall\ n > N$

(basis for τ)

By condition (3). Obviously $x \in V$

Let $y \in V \Rightarrow d(x,y) = \sum_{n=1}^{\infty} d_n(x_n, y_n)$

$$= \sum_{n=1}^{N} d_n(x_n, y_n) + \sum_{n=N+1}^{\infty} d_n(x_n, y_n)$$

$$< \sum_{n=1}^{N} \cancel{d_n(x_n, y_n)}\ \frac{r}{2N} + \sum_{n=N+1}^{\infty} d_n(x_n, y_n)$$

$$< \frac{r}{2N} \cdot N + \frac{r}{2} = r$$

$\Rightarrow d(x,y) < r$

$\Rightarrow y \in B(x, r)$, $\forall\ y \in V$

$\Rightarrow V \subset B(x, r)$, but $B(x, r) \subset G$

$\Rightarrow x \in V \subset G$

$\Rightarrow G$ is a nbd of each of its points

$\Rightarrow G$ is τ-open, i.e. $G \in \tau$.

$\Rightarrow \boxed{\mathcal{T}_d \subset \tau} \longrightarrow (4)$

($\Leftarrow$) Conv

Now we ~~have~~ show that $\tau \subset \mathcal{T}_d$.

let G be τ-open set in X containing x, without loss of generality, we may assume that $\exists$ a basis open set V s.t. $x \in V \subset G$, where $V = \prod_{n \in \mathbb{N}} V_n$

$V_n = X_n \ \forall\ n > N_1$ for some fixed and $V_n \subsetneq X_n$, $n = 1, 2, \ldots N$.

$\Rightarrow X_n - V_n$ is a closed set in X_n

we define $r_n = d_n(x_n, X_n - V_n)$, $n = 1, 2, \ldots N$ assume that $r \leq 2^{-n}$ $\forall\, n \in N$ and let $r = \min\{r_1, r_2, \ldots r_N\}$

By zorn's lemma. $r > 0$,

construct an open ball in X with centre at x and radius r, i.e. $B(x, r)$

claim - $B(x, r) \subset V$

Suppose $y = (y_n)_{n \in N} \in B(x, r)$, then $y \in B(x, r)$

$\Rightarrow d(x, y) < r$

$\Rightarrow \sum_{n=1}^{\infty} d_n(x_n, y_n) < r \Rightarrow d_n(x_n, y_n) < r \leq r_n$, $n = 1, 2, \ldots N$

$\Rightarrow y = \{y_n\} \in V$, but y was an arbitrary point of $B(x, r)$

$\Rightarrow B(x, r) \subset V$

$\Rightarrow B(x, r) \subset G$, since $V \subset G$

$\Rightarrow \exists\, r > 0$ such that $B(x, r) \subset G$

$\Rightarrow G$ is d-open , i.e. $G \in \mathcal{T}_d$

$\Rightarrow \tau \subset \mathcal{T}_d \longrightarrow (5)$

from eqn (4) and (5), we have

$\boxed{\tau = \mathcal{T}_d}$ proved.

K.D. (211)

Problem 1. $f : X \xrightarrow[\text{open}]{\text{continuous}} Y$. If X is first countable at $x_0 \in X$, then Y is first countable at $f(x_0)$

Problem 2. $f : X \xrightarrow[\text{open}]{\text{continuous}} Y$: If X is second countable then Y is second countable

OR

first / second countability is preserved under continuous open function

Proposition - Let $X = \prod_{i \in I} X_i$ and $x \in X$. Then X is I^C at $x \in X$ iff X_i is I^C at $\pi_i(x)$, $\forall\, i \in I$ and all except countably many i's, X_i is the only nbd of $\pi_i(x)$ in X_i.

Proof - Suppose that X is first countable at $x \in X$.
Let $x \in G$ be an open set in $X \Rightarrow \exists$ a countable local base $V^1, V^2, \ldots, V^n \ldots$ at x.

where $V^n = \prod_{i \in I} V_i^n$, $V_i^n = X_i$, $i \in I - J_n$, $\cancel{\ }J_n$ = a finite set

Let $J = \bigcup_{n=1}^{\infty} J_n$, then J is a countable subset of I.
We have to show that X_i is the only nbd of $\pi_i(x)$ for all $j \in I - J$ $(= I - J)$

If not, Then $\exists$ $j \in I - J$ and an open set G_j in X_j containing $\pi_j(x)$ such that $G_j \subsetneq X_j$.

Now Let $G = \pi_j^{-1}(G_j)$. Then G is an open set containing x.
So by the def^n of local base, $\exists\, n \in N$ s.t. $V^n \subset G$.

Now $J \notin J_n$ and so $X_j = V_j^n \subset G \Rightarrow \pi_j(V_j^n) \subset \pi_j(G) = G_j$
a contradiction. So our supposition is false.

Thus X_i is the only nbd of $\pi_i(x)$ in X_i

Conversely, Suppose each X_i is first countable at $\pi_i(x)$ for all $i \in I - J$, J being countable subset of I.
X_i is the only open set in X_i containing $\pi_i(x)$.

Let $\mathcal{F}_1$ be the set of all finite subset F of I
~~Then it is evident that $\mathcal{J} = \bigcup_{F \in \mathcal{F}_1} F$ for any~~
Since J is countable, so
Define

$\mathcal{F}_1 = \{ F \mid F \subset J$ and F is finite$\}$

$\mathcal{L}_F = \{ V \mid V = \prod_{i \in I} V_i,\ V_j \subsetneq X_j,\ V_j \in \mathcal{L}_j,\ j \in F \}$

where $\mathcal{L}_j$ denotes local base at $\pi_j(x)$ in X_j
Since each $\mathcal{L}_j$ is countable set and F is finite
F is countable.

Also each member of $\mathcal{L}_F$ is open nbd of x_0 in X.

Let $\mathcal{L} = \bigcup_{F \in \mathcal{F}_1} \mathcal{L}_F$. Then $\mathcal{L}$ is countable, since each $\mathcal{L}_F$ is countable and $\mathcal{F}_1$ is finite.

Claim - $\mathcal{L}$ is a local base at x_0 in X

Let G be open set in X containing x
Then we have to show that $\exists$ a $U \in \mathcal{L}$ s.t. $x \in U \subset G$

$\Rightarrow \exists$ a basis open set V s.t. $x \in V \subset G$

let $V = \prod_{i \in I} V_i$, note that for $i \neq J$, $V_i = X_i$,
since X_i is the only open set containing $\pi_i(x)$.
let $F = \{ j \in I : V_j \neq X_j \}$. Then F is a finite subset of I.

for each $j \in F$, $\exists$ $U_j \in F$ s.t. $U_j \subset V_j$.

set $U_j = X_j$, $\forall j \in I$, let $U = \prod_{i \in I} U_i$, Then U is an open nbd of x and $U \in \mathcal{L}_F$ s.t. $x \in U$

$\Rightarrow$ $x \in U \in \mathcal{L}$, moreover $U \subset V$ {by very construction}

$\Rightarrow$ $x \in U \subset V \longrightarrow (2)$

from eqn (1) and (2), we get

$x \in U \subset V \subset G$

Thus $\mathcal{L}$ is a local base of x ~~and~~

$\Rightarrow X$ is first countable at x. proved.

Pancham Patel

2017

Theorem – let $X = \prod_{i \in I} X_i$ is first countable at $x \in X$ iff each X_i (coordinate space) is first countable at $\pi_i(x)$ and for all except finitely many i's, X_i is indiscrete.

Proof. let $X = \prod_{i \in I} X_i$. Suppose X is first countable.

let $J = \{ i \in I \mid X_i$ is not indiscrete $\}$

claim – J is countable.

Suppose, if possible, J is finite then $\exists$ an index $i \in J$ s.t. $\exists$ a proper open subset G_i of X_i containing $\pi_i(x)$. Then it is evident that X is first countable at $x = (x_i)_{i \in I}$,

let $G_j = X_j$ for $j \in I - J$ and let $x \in \prod_{i \in I} G_i$

$x_i \in X_i$ $\forall i \notin J$, $x_i \in G_i$ $\forall i \in J$,

But there are uncountably many indices j for which X_j is not only nbd of $\pi_j(x)$ in X_j.

$\Rightarrow$ a contradiction to, let $X = \prod_{i \in I} X_i$ is I^c at $x \in X$ iff for each $i \in I$, X_i is I^c at $\pi_i(x)$ and for all except countably many i's, X_i is the only nbd of $\pi_i(x)$ in X_i

$\Rightarrow$ All except ~~countably many~~ finitely many i's, X_i is indiscrete.

conversely $\rightarrow$ (*) from last proposition.

Theorem – Let $X = \prod_{i \in I} X_i$ is second countable iff each X_i is second countable and for all except countably many i's, X_i is indiscrete space.

Proof: Suppose $X = \prod_{i \in I} X_i$ is second countable.

Since $\pi_i : X \longrightarrow X_i$, which is known to be continuous and open, and also second countable is preserved under continuous open function. Hence each X_i is second countable.

Moreover, ~~sin~~ X is also first countable, but $X = \prod_{i \in I} X_i$ is I^c iff each X_i is I^c and all except finitely many i's, X_i is indiscrete.

So we conclude that, X_i is Π^c and all except countably many i's; X_i is indiscrete space.

Conversely – Suppose each X_i is second countable and the set

$J = \{ i \in I : X_i$ is not indiscrete$\}$ is countable

Let $\mathcal{B}_j$ be a countable base for X_j for $j \in J$. "$\mathcal{B}_j$ is a basis for topology $(X_j, \mathcal{T}_j)$". For a finite subset F of J. Let $\mathcal{B}_F$ be the collection of all large boxes with 'short' sides lies in $\mathcal{B}_j$ for $j \in F$

i.e. $\mathcal{B}_F = \{ \prod_{i \in I} Y_i : Y_i = X_i$ for $i \notin F$ & $Y_j \in \mathcal{B}_j$ for $j \in F \}$

Since each $\mathcal{B}_j$ is countable and F is finite, $\mathcal{B}_F$ is countable. Let $\mathcal{F}$ be the collection of all finite subset J and let

$$\mathcal{B} = \bigcup_{F \in \mathcal{F}} \mathcal{B}_F$$

Observation –

(I) $\mathcal{B}$ is a countable.

(II) $\mathcal{F}$ is countable, because it is countable collection of finite sets.

(III) $\mathcal{B}_F$ is countable, —— " —— " ——

(IV) Each member of $\mathcal{B}$ belong to the standard base for the product topology and hence is an open set.

$\Rightarrow$ $\mathcal{B}$ is countable basis for $\mathcal{T}$

$\Rightarrow$ $\mathcal{B}$ is a countable base for every $\mathcal{T}$

$\Rightarrow$ X is second countable.

proved.

पुंरदाम पटेल

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QUE I (35)

Theorem – If $\mathcal{B}$ and $\mathcal{C}$ are respectively basis of $\mathcal{T}_1$ and $\mathcal{T}_2$, then the collection $\mathcal{A} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$ is a basis for product topology on $X \times Y$.

Proof – Given

QUE II (36) Suppose $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ are projection maps, Then

$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \in \mathcal{T}_1\} \cup \{\pi_2^{-1}(V) \mid V \in \mathcal{T}_2\}$ is a subbase for product topology on $X \times Y$.

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Evaluation function – Let $\{Y_i : i \in I\}$ be an indexed family of sets. Suppose X is a set and let for each $i \in I$, $f_i : X \longrightarrow Y_i$ be a function. Then the function $e : X \longrightarrow \prod_{i \in I} Y_i$ defined by $e(x)(i) = f_i(x)$ $\forall\, i \in I, x \in X$ is called the evaluation function of the indexed family $\{f_i : i \in I\}$ of functions.

Proposition – Let $\{Y_i : i \in I\}$ be a family of sets, X a set and for each $i \in I$, $f_i : X \to Y_i$ a function. Then the evaluation function is the only function from X into $\prod Y_i$ whose composition with the projection $\pi_i : \prod Y_i \mapsto Y_i$ equals f_i $\forall\, i \in I$.

Proof – Let $e : X \longrightarrow \prod_{i \in I} Y_i$ be the evaluation function of the family $\{f_i : i \in I\}$. Then ~~for~~ by definition of e, $\pi_i(e(x)) = e(x)(i) = f_i(x) \longrightarrow (1)$
and so $\pi_i \circ e = f_i$

Suppose, if possible $\exists$ another evaluation $e' : X \longrightarrow \prod_{i \in I} Y_i$.
then $e'(x)(i) = f_i(x) \longrightarrow (2)$

from equation (1) and (2), we have

$$e(x)(i) = e'(x)(i)$$

because i is arbitrary,

$$e(x) = e'(x) \quad \forall\, x \in X$$

$$\Rightarrow \boxed{e = e'} \longrightarrow (3)\ \forall\, x \in X$$

Again $e(x)(i) = f_i(x) \quad \forall\, x \in X$

$\Rightarrow \pi_i(e(x)) = f_i(x) \quad \forall\, x \in X$

$\Rightarrow (\pi_i \circ e)(x) = f_i(x) \quad \forall\, x \in X$

$\Rightarrow \boxed{\pi_i \circ e = f_i}$

Theorem - Show that e is continuous iff each f_i is continuous.

Proof - Suppose e is continuous. Since each projection function π_i is continuous for each i.

$\Rightarrow$ the composite function $\pi_i \circ e$ is continuous but $\pi_i \circ e = f_i \quad \forall\, i \in I$

$\Rightarrow$ each f_i is continuous.

Conversely, suppose each f_i is continuous, but every projection function is continuous, so π_i is continuous.

$$f_i = \pi_i \circ e \quad \forall\, i \in I$$

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$$\Rightarrow f_i(x) = (\pi_i \circ e)(x) \quad \forall\, x \in X$$

Assume that f_i is continuous $\forall\, i \in I$ and $\pi_i^{-1}(A)$ is a subbasis open subset of $\prod Y_i$, where A is open in A_i, we have $\pi_i \circ e = f_i \; \forall\, i \in I$ so that $e^{-1}[\pi_i^{-1}[A]] = (e^{-1} \circ \pi_i^{-1})[A] = (\pi_i \circ e)^{-1}[A] = f_i^{-1}[A] \Rightarrow e^{-1}[\pi_i^{-1}[A]]$ is open in X, since f_i is open cont. & $f_i^{-1}[A]$ is open in $X \Rightarrow e^{-1}[A]$ is open in X and hence e is continuous.

Distinguish points - An indexed family $\{f_i : i \in I\}$ of functions all defined on the same domain X is said to be distinguish points if for any distinct $x, y \in X$, $\exists\, j \in I$ such that

$$f_j(x) \neq f_j(y).$$

Proposition - e is one-one iff $\{f_i\}_{i \in I}$ distinguish points.

Proof - Let $f_i : X \to Y_i$ be a function for $i \in I$ and let $e : X \to \prod Y_i$ be the evaluation function.

Let x, y be distinct points of X. Then, $e(x) \neq e(y)$

~~$e(x) \neq e(y)$ iff $\exists$~~

But by definition of e, $e(x)(j) = f_j(x)$

similarly, $e(y)(j) = f_j(y)$

But $e(x) \neq e(y) \Rightarrow e(x)(j) \neq e(y)(j) \quad \forall\, j \in I$

$\Rightarrow \boxed{f_j(x) \neq f_j(y)} \quad \forall\, x, y \in X$

Conversely, let $x, y \in X$ s.t. $x \neq y$, then $f_j(x) \neq f_j(y)$

$\Rightarrow e(x)(j) \neq e(y)(j) \quad \forall\, j \in I$

$\Rightarrow \boxed{e(x) \neq e(y)} \quad \forall\, x, y \in X.$

Com

Distinguish points – The family of function $\{f_i \mid f_i : X \to Y_i,\ i \in I\}$ is said to be distinguish points from closed sets, if for $x \in X$ and a closed set C not containing x, $\exists\ j \in I$ such that $f_j(x) \notin \overline{f_j(C)}$ in Y_j.

Proposition – A T.S. X is completely regular iff the family of all real-valued continuous function defined on X it distinguish points from closed sets.

Imp. Proof

Proof – ($\Rightarrow$) Suppose X is completely regular, let $x \in X$ and C is a closed set in X not containing x.
Suppose $\mathcal{F}$ denotes the family of all real-valued continuous functions on X.
i.e. $\mathcal{F} = \{f_i \mid f_i : X \xrightarrow{\text{continuous}} R\}$.
Then $\exists$ ~~a con~~ by complete regularity of X, $\exists$ a continuous funⁿ $f : X \to [0,1]$ s.t. $f(x) = 0$ and $f(C) \subseteq \{1\}$. (if $C = \phi$, then $f(C) = \{1\}$)

Define an inclusion map $i : [0,1] \to R$ by $i(x) = x$ $\forall\ x \in [0,1]$
Then $i \circ f : X \to R$ such that
(i) $i \circ f$ is continuous
(ii) $(i \circ f)(x) = i(f(x)) = f(x)$ $\forall\ x \in X$

Thus $i \circ f \in \mathcal{F}$ s.t. $0 = f(x) = (i\circ f)(x) \notin \overline{(i\circ f)(C)}$ $\{\overline{\{1\}} = \{1\}\}$
$\Rightarrow f(x) \notin \overline{f(C)}$
$\Rightarrow \mathcal{F}$ distinguishes points from closed set $\}$.

Conversely, suppose $\mathcal{F}$ distinguishes points from closed set
$\Rightarrow \exists\ f \in \mathcal{F}$ such that $f(x) \notin \overline{f(C)}$

$\Rightarrow \{f(x)\}$ and $\overline{f(C)}$ are disjoint closed set in R and that R is a normal space.

$\Rightarrow$ By Urysohn's lemma, $\exists$ a continuous function $g : R \to [0,1]$ such that
$\{(g\circ f)(x)\} = \{g(f(x))\} = 0$
$g(\overline{f(C)}) = \{1\}$
Let $h : X \to [0,1]$ be the composite $g\circ f$.
Then $h(x) = 0$; $\overline{h(C)} = \{1\}$
$\Rightarrow h(x) \notin \overline{h(C)}$ such that $h \in \mathcal{F}$

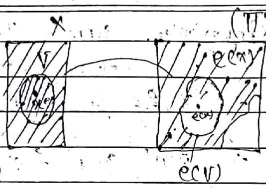
$\Rightarrow X$ is completely regular.

Embedding lemma (uniform topology) Imp

Proposition – Let ~~$\{f_i$~~ The family of function $\{f_i \mid f_i : X \to Y_i\}$ distinguishes points from closed sets in X, then the evaluation function $e : X \to \prod Y_i$ is open when e is regarded as a function from X onto $e(X)$.

Proof – Let V be an open set in X. Then we have to show that $e(V)$ is an open subset of $e(X)$.

Let $x \in V \subset X$, V being open
$\Rightarrow X - V$ is a closed set not containing x.

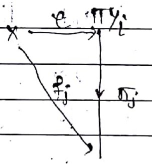


So the family of continuous real-valued function $\mathcal{F} = \{f_i \mid f_i : X \xrightarrow{\text{continuous}} Y_i\}_{i \in I}$ distinguish points from closed sets, ~~entw/s~~. $\exists\ j \in I$ s.t.

$$f_j(x) \notin \overline{f_j(X-V)}$$

Let $G_j = Y_j - \overline{f_j(X-V)}$, Then G_j is an open set in Y_j containing $f_j(x)$

<u>claim</u> - $e(x) \in G \cap e(X) \subset e(V)$



Let $y \in G \cap e(X) \Rightarrow y \in G$ and $y \in e(X)$

$\Rightarrow \exists\ z \in X$ such that $y = e(z)$, $\pi_j^{-1}(y) \in G_j$ $(G_j = \pi_j(G))$

$\Rightarrow \pi_j(e(z)) \in G_j$

$\Rightarrow f_j(z) \in G_j \longrightarrow$ (1)

Now we show $z \in V$, if not, then $z \in X - V$

$\Rightarrow f_j(z) \in f_j(X-V) \subset \overline{f_j(X-V)} = Y_j - G_j$

a contradiction to $f_j(z) \in G_j$

Thus $z \in V \Rightarrow y = e(z) \in e(V)$

~~Now~~ $\Rightarrow G \cap e(X) \subset e(V)$

$\Rightarrow f_j(x) \in G_j = \pi_j(G)$

$\Rightarrow \pi_j^{-1}(f_j(x)) \in \pi_j^{-1}(\pi_j(G))$

$\Rightarrow \pi_j^{-1}(f_j(x)) \in G$

$\Rightarrow e(x) \in G$

Thus $e(x) \in G \cap e(X) \subset e(V)$

$\Rightarrow e(V)$ is a nbd of each of its points

$\Rightarrow e(V)$ is open in $e(X)$. <u>proved.</u>

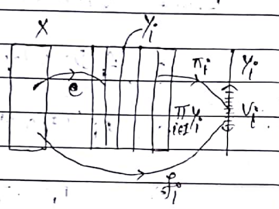
Imp

Lemma :-

<u>Embedding Theorem</u> - Suppose $\{f_i \mid f_i : X \longrightarrow Y_i\}_{i \in I}$ is the given family of continuous function having the properties that is distinguishe points and also distinguishes points from closed set. Then the corresponding evaluation function is an embedding from X into $\prod_{i \in I} Y_i$

<u>Proof</u> - Let $e : X \longrightarrow \prod_{i \in I} Y_i$ be the evaluation function. To show that it is an embedding, we have to show that e is an open, continuous bijection when regarded as a function from X to $e(X)$.

(I) e is one-one - As the family of function $\{f_i\}_{i \in I}$ distinguishes points, it follows that the corresponding evaluation function $e : X \longrightarrow \prod_{i \in I} Y_i$ is one-one.



(II) e is open - Again the family $\{f_i\}_{i \in I}$ distinguish points from closed sets, it follows that the evaluation function e is open.

(III) e is continuous :- We have

$$\pi_i \circ e = f_i \quad \forall \ i \in I$$

Suppose V_i is an open set in Y_i, then

$$\pi_i^{-1}(V_i) = \prod_{j \in I \setminus \{i\}} X_j \times V_i \quad \{= \text{or } X_1 \times X_2 \times \cdots \times X_{i-1} \times V_i \times V_{i+1} \times \cdots\}$$

Since $\pi_i : \prod_{i \in I} Y_i \longrightarrow Y_i$ is continuous

$\Rightarrow$ $\pi_i^{-1}(V_i)$ is open in $\prod_{i \in I} Y_i$, of course it is a subbasic open set in $\prod_{i \in I} Y_i$.

Now $f_i^{-1}(V_i)$ is open in X, by continuity of f_i

$\Rightarrow$ $(e^{-1} \circ \pi_i^{-1})(V_i) = (\pi_i \circ e)^{-1}(V_i)$ is open in X

$\Rightarrow$ $e^{-1}(\pi_i^{-1}(V_i))$ is open in X

$\Rightarrow$ e is continuous

(IV) e is onto - In view of (II), e should be regarded as a function from X onto e(X).

$\Rightarrow$ $\exists$ a homeomorphism e from X onto a subspace $(e(X), \tau/e(X))$ of $(\prod_{i \in I} Y_i, \tau)$.

So we conclude that e is an embedding ~~of~~ from X into $\prod_{i \in I} Y_i$.

imp

Theorem

Tychonoff Embedding lemma - (273 H.K)

Statement - A topological space is a tychonoff space if and only if it is embeddable into a cube.

Proof - ($\Leftarrow$) Suppose the given topological space X is a tychonoff space, then we have to show that it can be embedded into a cube.

Let $\mathcal{F}$ denote the family of all continuous function from X into [0,1]

i.e. $\mathcal{F} = \{ f : X \longrightarrow [0,1] \}$ be continuous.

Consider the cube $[0,1]^{\mathcal{F}}$, we intend to show that X can be embedded into $[0,1]^{\mathcal{F}}$

Since X is tychonoff $\Rightarrow$ X is completely regular as well

⇒ The family of function $\mathcal{F}$ distinguish point from closed sets.

and by virtue of T_1- axiom, the family $\mathcal{F}$ also distinguish points.

By the proposition which states that "let $\{f_i \mid f_i : X \to Y_i\}_{i \in I}$ be the given family of continuous function having the properties that is distinguish points and distinguish points from closed sets, then the corresponding evaluation function is an embedding from X into $\prod_{i \in I} Y_i$." thus the corresponding evaluation function

$$e : X \longrightarrow [0,1]^{\mathcal{F}}$$

will be instrumental to embed X into the cube $[0,1]^{\mathcal{F}}$.

Conversely -

(⇒) We know that every cube is a Tychonoff space. But the tychoff property is hereditary. so every subspace of a cube and every space homeomorphic to a subspace of a cube is a Tychonoff space.

$[0,1]^{I \to \text{countable}}$ Hilbert cube

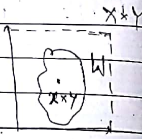
$[0,1]^{I \to \text{uncountable}} \Rightarrow$ cube.

Problem 1- If $(X, \mathcal{T})$ and $(Y, \mathcal{V})$ are two topological space with respective basis $\mathcal{B}$ and $\mathcal{C}$, then show that

$$\mathcal{B} \times \mathcal{C} = \{ B \times C \mid B \in \mathcal{B}, C \in \mathcal{C} \}$$

is a base for product topological space $X \times Y$.

Solution - let τ be the product topology on $X \times Y$. Then by very definition the collection $\{U \times V \mid U \in \mathcal{T}, V \in \mathcal{V}\}$ is the basis open sets.



let $x \times y$ be a typical point of W an open set in $X \times Y$. Then by definition, $\exists$ a basis open set, say $U \times V$ s.t. $x \times y \in U \times V \subset W \longrightarrow$ (1)

⇒ $x \in U$, $y \in V$

⇒ $\exists$ a $B \in \mathcal{B}$ and a $C \in \mathcal{C}$ s.t. $x \in B \subset U$, $y \in C \subset V$

⇒ $\exists$ a $B \times C \in \mathcal{B} \times \mathcal{C}$ s.t. $x \times y \in B \times C \subset U \times V \subset W$ {by (1)}

⇒ $\mathcal{B} \times \mathcal{C}$ is basis for τ.

proved.

पंचशील पटेल

(P.patel)

Problem 2. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are defined by $\pi_1(x \times y) = x$ and $\pi_2(x \times y) = y$, Then

$$\mathcal{S} = \{ \pi_1^{-1}(U) \mid U \in \mathcal{T} \} \cup \{ \pi_2^{-1}(V) \mid V \in \mathcal{V} \}$$

is a subbase for τ (product topology).

Proof - Let τ be the product topology on $X \times Y$. Then the collection $\{ U \times V \mid U \in \mathcal{T}, V \in \mathcal{V} \}$ is the basic open set.

Claim - $\tau = \tau'$, where τ' being the topology generated by $\mathcal{S}$ i.e. the smallest topology containing $\mathcal{S}$.

Observation - Every element of $\mathcal{S}$ is an open set in $X \times Y$ w.r.t. to product topology on $X \times Y$. $\mathcal{S} \subset \tau$. But τ is closed w.r. to finite intersection and all union.

$$\Rightarrow \quad \tau' \subset \tau \longrightarrow (1)$$

In order to show that $\tau \subset \tau'$, it is just require to show that $\mathcal{B} \subset \mathcal{B}'$

$\mathcal{S}$

Locally finite - Let X be a topological space. A collection ~~$\mathcal{A}$~~ $\mathcal{A}$ of subset of X such that for each $x \in X$, there is a neighbourhood which intersects only finite members of ~~$\mathcal{A}$~~. $\mathcal{A}$.

Ex (1) The collection of intervals

$\mathcal{A} = \{ (n, n+2) \mid n \in \mathbb{Z}_+ \}$ is locally finite in the topological space $\mathbb{R}$

But $\mathcal{B} = \{ (0, 1/n) \mid n \in \mathbb{Z}_+ \}$ is locally finite in $(0,1)$, but not in $\mathbb{R}$.

Munkres
Lemma - Let $\mathcal{A}$ be a locally finite collection of
§45 subset of X, then

(I) Any subcollection of $\mathcal{A}$ is locally finite.
(II) $\{ \overline{A} \mid A \in \mathcal{A} \}$ is locally finite.
(III) $\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

Proof (I) Suppose $\mathcal{B}$ is a subcollection of $\mathcal{A}$, For each $x \in X$, $\exists$ a nbd which intersects a finite members of $\mathcal{A}$ $\Rightarrow$ N_x will ~~be~~ do intersects either with the same number of members of $\mathcal{B}$ or else with fewer number of members of $\mathcal{B}$.

$\Rightarrow \mathcal{B}$ is locally finite.

(II) Note that any open set U that intersects the set $\bar{A}$ necessary intersects A, Therefore if U is a nbd of x that intersects only finitely many elements ~~of~~ A of $\mathcal{A}$, then U can intersects atmost the same numbers of sets of the collection $\bar{A}$. Hence $\{\bar{A}\}_{A\in\mathcal{A}}$ is the closure of elements of $\mathcal{A}$ is locally finite.

(III) Set $Y = \bigcup_{A\in\mathcal{A}} A$, then $A \subset Y$ $\forall A \in \mathcal{A}$

$\Rightarrow \bar{A} \subset \bar{Y}$ $\forall A \in \mathcal{A}$

$\Rightarrow \bigcup_{A\in\mathcal{A}} \bar{A} \subset \bar{Y}$

$\Rightarrow \bigcup_{A\in\mathcal{A}} \bar{A} \subset \overline{\bigcup_{A\in\mathcal{A}} A} \longrightarrow (1)$

Now let $y \in \bar{Y} = \{y \in X, U \in \eta_y, U \cap Y \neq \phi\} \Rightarrow U \cap Y \neq \phi$

Suppose $A_1, A_2, \ldots, A_k \in \mathcal{A}$ intersect U for some nbd U of y

We now claim that y is in some members among $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_k$

If, not, then ~~$y \in \bar{A}_i$~~ $y \in U - \bar{A}_1 - \bar{A}_2 - \cdots - \bar{A}_k$ and it is a nbd of y which do not intersect any members of $\mathcal{A}$ which contradicts to local finiteness of $\mathcal{A}$

~~now y~~

$\Rightarrow y$ belong to some members among $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_k$

$\Rightarrow y \in \bigcup_{i=1}^{k} \bar{A}_i \subset \bigcup_{A\in\mathcal{A}} \bar{A}$

$\Rightarrow \bar{Y} \subset \bigcup_{A\in\mathcal{A}} \bar{A}$

$\Rightarrow \overline{\bigcup_{A\in\mathcal{A}} A} \subset \bigcup_{A\in\mathcal{A}} \bar{A} \longrightarrow (2)$

from eq^n (1) and (2) we have

$\overline{\bigcup_{A\in\mathcal{A}} A} = \bigcup_{A\in\mathcal{A}} \bar{A}$ proved.

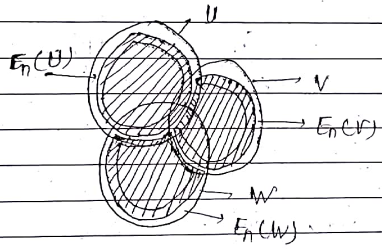
Countably locally finite - A collection $\mathcal{B}$ of subset of X is said to be countably locally finite if $\mathcal{B}$ can be expressed as countable union of locally finite families.

i.e. $\mathcal{B} = \bigcup_{n\in\mathbb{Z}_+} \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite.

Refinement - Let $\mathcal{A}$ be a collection of subset of the space X. A collection $\mathcal{B}$ of subset of X is said to be refinement of $\mathcal{A}$. If for each $B \in \mathcal{B}$, $\exists$ a $A \in \mathcal{A}$ such that $A \supset B$ or $\boxed{B \subset A}$

Lemma 39.2 (246) Let X be a metrizable space. If $\mathcal{A}$ be an open cover of X, then $\exists$ an open refinement $\mathcal{E}$ of $\mathcal{A}$ which covers X and that each member of $\mathcal{E}$ is a ~~loc~~ locally finite.

Proof -



Let us denote the element of $\mathcal{A}$ generically by the letters U, V, W such that $U < V < W < \dots$

choosing be a metric for X. let n be a positive integer, fixed for the moment. Given an element U of $\mathcal{A}$, let us define $S_n(U)$ to be the subset of U obtained by "shrinking" U

a distance of $1/n$.

let $S_n(U) = \{x \mid B(x, 1/n) \subset U\}$

Now we use the well-ordering $<$ of $\mathcal{A}$ to pass to a still smaller set. for each U in $\mathcal{A}$, define

$$T_n(U) = S_n(U) - \bigcup_{Z<U} Z$$

Similarly,

$$S_n(V) = \{x \in V \mid B(x, 1/n) \subset V\}$$

$$\Rightarrow T_n(V) = S_n(V) - \bigcup_{Y<V} Y$$

and

$$S_n(W) = \{x \in W \mid B(x, 1/n) \subset W\}$$

$$\Rightarrow T_n(W) = S_n(W) - \bigcup_{X<W} X$$

Now to show that $d(x,y) \geq 1/n$ $\forall$ $x \in T_n(U)$, $y \in T_n(V)$

let $U < V$, since $x \in T_n(U) \Rightarrow x \in S_n(U)$

and $y \in T_n(V) \Rightarrow y \in S_n(V)$

Since $U < V$, by definition of the latter set tells us that $y \notin U$

So we conclude that $d(x,y) \geq 1/n$ $\forall$ $x \in T_n(U)$, $y \in T_n(V)$

Let $E_n(U) = \bigcup_{x \in T_n(U)} B(x, 1/3n)$

$E_n(V) = \bigcup_{x \in T_n(V)} B(x, 1/3n)$

and $E_n(W) = \bigcup_{x \in T_n(W)} B(x, 1/3n)$

Since $x \in T_n(U)$, $y \in T_n(V) \Rightarrow d(x,y) \geq 1/n$

Similarly, $x \in E_n(U)$, $y \in E_n(V) \Rightarrow d(x,y) \geq 1/3n$

~~Claim~~ - Now let us define

$\mathcal{E}_n = \{ E_n(U) \mid U \in \mathcal{A} \}$

claim - $\mathcal{E}_n$ is an open refinement of $\mathcal{A}$ of course it will be locally finite.

(i) $\mathcal{E}_n$ is locally finite because $\exists$ a nbd $N = B(x, 1/6n)$ of x which intersects only one members of $\mathcal{E}_n$.

(ii) $\mathcal{E} = \bigcup_{n \in Z_+} \mathcal{E}_n$. Then we see that $\mathcal{E}$ is countably locally finite.

(iii) $\mathcal{E}$ covers X.

verification - let $x \in X$, let U be the first member of $\mathcal{A}$ containing x, then we can find for $n \in Z_+$ such that

$B(x, 1/n) \subset U$

$\Rightarrow x \in S_n(U)$

$\Rightarrow x \in T_n(U)$

$\Rightarrow x \in E_n(U)$

$\Rightarrow x \in E_n(U) \in \mathcal{E}_n \subset \mathcal{E}$

$\Rightarrow \mathcal{E}$ covers X. proved.

G_δ - set :- A subset A of a space X is called a G_δ - set in X, if it equals the intersection of a countable collection of open subsets of X.

i.e. $A = \bigcap_{n \in Z_+} U_n \rightarrow$ open.

F_σ - set - A subset A of a space X is called a F_σ set in X, if it equals the union of a countable collection of closed subsets of X.

i.e. $A = \bigcup_{n \in Z_+} F_n \rightarrow$ closed.

EX. (I) Each open set G in X is a G_δ - set.

(II) If X is first-countable Hausdorff, then each singleton set in X is a G_δ - set.

(*) (III) In a metric space X, each closed set a G_δ-set. Given $A \subset X$ let $U(A, \epsilon)$ denote the ϵ-nbd of A. If A is closed,

$$A = \bigcap_{n \in Z_+} U(A, 1/n)$$

(finite intersection of open set is closed).

Lemma - Let X be a regular space having a basis $\mathcal{B}$ that is countably locally finite. Then X is normal, and every closed set in X is a G_δ set in X.

Proof -

Step I. Let W be open in X. $\exists$ a countable collection of open sets $\{U_n\}$ such that

$$W = \bigcup_{n \in Z_+} U_n = \bigcup_{n \in Z_+} \overline{U_n}$$

Since the basis $\mathcal{B}$ for X is countably locally finite

$\Rightarrow$ $\mathcal{B} = \bigcup_{n \in Z_+} \mathcal{B}_n$, where each $\mathcal{B}_n$ is locally finite.

Let for a fixed n, choose $B \in \mathcal{B}_n$ ~~and~~ s.t. $\overline{B} \subset W$

$$\mathcal{C}_n = \{ B \in \mathcal{B}_n \mid \overline{B} \subset W \} \subset \mathcal{B}$$

Then $\mathcal{C}_n$ is locally finite.

Define $U_n = \bigcup_{B \in \mathcal{C}_n} B \quad \forall\, n \in Z_+ \longrightarrow (1)$

Then U_n is an open set.

$\overline{U_n} = \bigcup_{B \in \mathcal{C}_n} \overline{B}$ { by lemma 39.1, Munk. 249 }

Now $\overline{B} \subset W \quad \forall\, B \in \mathcal{C}_n$

$\Rightarrow \bigcup_{B \in \mathcal{C}_n} \overline{B} \subset W$

$\Rightarrow \overline{U_n} \subset W$

$\Rightarrow U_n \subset \overline{U_n} \subset W \quad \forall\, n$.

$\Rightarrow \boxed{\bigcup_{n \in Z_+} U_n \subset \bigcup_{n \in Z_+} \overline{U_n} \subset W} \longrightarrow (2)$

Claim - $W \subset \bigcup_{n \in Z_+} \overline{U_n}$

For this, let $x \in W$. there is by regularity a basis element $B \in \mathcal{B}_n$ s.t. $x \in B$ and $\overline{B} \subset W$

Now $B \in \mathcal{B}_n$ for some n, then $B \in \mathcal{C}_n$ s.t. $x \in B$

$\Rightarrow x \in U_n = \bigcup_{B \in \mathcal{C}} B \quad \forall\, n$.

$\Rightarrow x \in \bigcup_{n \in Z_+} U_n$

$\Rightarrow W \subset \bigcup U_n \subset \bigcup \overline{U_n} \longrightarrow (3)$

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from eqn (7) & (8)

$$\boxed{W = \bigcup_{n \in Z_+} U_n = \bigcup_{n \in Z_+} \overline{U_n}}$$

Step II. Every closed set C in X is a G_δ-set in X.

For this, Given C, let $W = X - C$ by step I

$$X - C = W = \bigcup_{n \in Z_+} \overline{U_n}$$

$$\Rightarrow C = X - W$$

$$\Rightarrow C = X - \bigcup_{n \in Z_+} \overline{U_n}$$

$$\Rightarrow C = \bigcap_{n \in Z_+} (X - \overline{U_n})$$ = countable intersection of open sets

$\Rightarrow$ C is a G_δ-set in X.

Step III. X is normal - Let C and D be disjoint closed set in X, i.e. $C \cap D = \phi$

Applying step (I) to the open set

$$X - D = \bigcup_{n \in Z_+} U_n = \bigcup_{n \in Z_+} \overline{U_n}$$

$\Rightarrow C \subset X - D = \bigcup_{n \in Z_+} U_n$, $D \cap \overline{U_n} = \phi$ $\forall n \in Z_+$

Similarly, $X - C = \bigcup_{n \in Z_+} V_n = \bigcup_{n \in Z_+} \overline{V_n}$ such that

$D \subset X - C = \bigcup_{n \in Z_+} V_n$, $C \cap \overline{V_n} = \phi$ $\forall n \in Z_+$

Then $\{U_n\}$ covers C and each set $\overline{U_n}$ is disjoint from D. Similarly there is a countable covering $\{V_n\}$ of D by open sets whose closures are disjoint from C.

Define $U_n' = U_n - \bigcup_{i=1}^{n} \overline{V_i}$ and $V_n' = V_n - \bigcup_{i=1}^{n} \overline{U_i}$

Then the sets

$$U' = \bigcup_{n \in Z_+} U_n' \text{ and } V' = \bigcup_{n \in Z_+} V_n'$$

are disjoint open sets about C and D respectively.

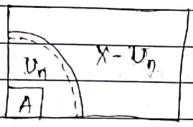
Imp. MCQ's

Theorem 40.2 Suppose X is a normal space, let A be a closed G_δ set in X. Then there is a continuous function $f: X \to [0,1]$ such that $f(x) = 0$ $\forall x \in A$ and $f(x) > 0$ for $x \notin A$.

Proof - Let A be a G_δ-set. $\exists$ a family of open sets $\{U_n\}$ such that

$A = \bigcap_{n \in Z_+} U_n \Rightarrow A \subset U_n$ for each $n \in Z_+$.

clearly, A and $X - U_n$ are disjoint closed set in X and that X is normal.



By Urysohn's lemma for each n, $\exists$ a continuous function

$$f_n : X \longrightarrow [0,1] \quad \text{such that}$$

$f_n(x) = 0 \ \forall \ x \in A$, $f_n(x) = 1 \ \forall \ x \in X - U_n$ i.e. outside of U_n

Define $$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}$$

The series converges uniformly by comparison with $\Sigma \frac{1}{2^n}$.

So that f is continuous. Also f vanishes on A and it is positive on $X-A$

i.e. $f(x) = 0 \ \forall \ x \in A$ and $f(x) > 0 \ \forall \ x \notin A$.

proved.

2010, 14, 16

Theorem (Nagata-Smirnov metrization theorem):-

Imp

A space X is metrizable if and only if X is regular and has a basis that is countably locally finite.

Proof - ($\Leftarrow$) "If condition" - Suppose X is regular having a base $\mathcal{B}$ which is countably locally finite. Then X is normal and every closed set in X is a G_δ set in X.

We shall show that X is metrizable by embedding X in the metric space $(\mathbb{R}^J, \bar{\rho})$ for some J.

Let $\mathcal{B} = \bigcup_{n \in \mathbb{Z}_+} \mathcal{B}_n$, where each $\mathcal{B}_n$ is locally finite. For fixed n, choose a $B \in \mathcal{B}_n$ and define a continuous function

$$f_{n,B} : X \longrightarrow [0, 1/n] \quad \text{such that}$$

$$f_{n,B}(x) > 0 \ \forall \ x \in B \quad \text{and} \quad f_{n,B}(x) = 0 \ \forall \ x \notin B$$

The collection $\{f_{n,B}\}$ separates points from closed sets in X. Given (in X) a point x_0 and a neighbourhood U of x_0, there is a basis element B s.t. $x_0 \in B \subset U$. Then $B \in \mathcal{B}_n$ for some n, so that

$$f_{n,B}(x_0) > 0 \quad \text{and} \quad f_{n,B}(x_0) = 0 \quad \forall \ x_0 \notin U.$$

$\forall x_0 \in B$

Let J be the subset of $Z_+ \times \mathcal{B}$ consisting of all pairs (n, B) s.t. $B \in \mathcal{B}_n$

Define $F: X \to [0,1]^J$ by the equation

$$F(x) = (f_{n,B}(x))_{(n,B) \in J}.$$

Relative to the product topology on $[0,1]^J$. The map F is an imbedding. "by imbedding lemma"

"Imbedding lemma - (i) each singleton set is closed.
(ii) $\exists$ a continuous function from $\{f_\alpha \mid f_\alpha : X \to \mathbb{R}\}$ which separates points from closed sets. Then X can be embedded into $(R^J, \bar{\rho})$ equipped with uniform metric $\bar{\rho}$, i.e. function $F: X \to \mathbb{R}$ defined by

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is an imbedding of X in R^J. In particular F imbeds in $[0,1]^J$ equipped with ρ being uniform metric on $[0,1]^J$."

i.e. $([0,1]^J, \mathcal{T}_\rho)$ is a topological space
The uniform topology is finer than the product topology $(\mathcal{T} \subset \mathcal{T}_\rho)$

$\therefore$ relative to the uniform metric, the map

(i) F is injective (ii) F is open

(iii) F is regarded as a funⁿ from X onto $Z = F(X)$
(iv) F is continuous.

Note that on the subspace $[0,1]^J$ of R^J, the uniform metric equals the metric $(\rho = \bar{\rho})$

$$\rho((x_\alpha), (y_\alpha)) = \sup\{|x_\alpha - y_\alpha|\}$$

To prove continuity, let $x_0 \in X$ and $\epsilon > 0$, and find a nbd W of x_0 s.t.

$$x_0 \in W \Rightarrow \rho(F(x), F(x_0)) < \epsilon$$

Let n be fixed for the moment, choose a nbd U_n of x_0 that intersects only finitely many elements of the collection $\mathcal{B}_n$. each function $f_{n,B}$ is continuous we can now choose an open set V_n of x_0 contained in U_n $\forall n \in Z_+$ s.t. $\epsilon > 0$.

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq \epsilon/2 \quad \forall x \in V$$

choose $N \in Z_+$ s.t. $1/N \leq \epsilon/2$, and ~~define~~ Take $W = V_1 \cap V_2 \cap \dots \cap V_N$
W is the desired nbd of x_0.

Let $x \in W$. If $n \leq N$ then

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq 1/n \leq \epsilon/2$$

because $f_{n,B}$ maps X into $[0, 1/n]$. Therefore

$\rho(F(x), F(x_0)) \le \varepsilon/2 < \varepsilon$

case (ii) for $x \in X$, $n > N \Rightarrow |f_{n,B}(x) - f_{n,B}(x_0)| \le \frac{1}{n} < \frac{1}{N} \le \frac{\varepsilon}{2}$

~~(ii) conversely,~~

Thus for any $n \in \mathbb{Z}_+$ and $x \in W$

$\Rightarrow |f_{n,B}(x) - f_{n,B}(x_0)| < \varepsilon/2$

$\Rightarrow \sup_{(n,B) \in J} |f_{n,B}(x) - f_{n,B}(x_0)| \le \varepsilon/2$

$\Rightarrow \rho((f_{n,B}(x)), (f_{n,B}(x_0))) \le \varepsilon/2 < \varepsilon$

$\Rightarrow \rho(F(x), F(x_0)) < \varepsilon \quad \forall\, x \in X$

$\Rightarrow$ F is continuous at x_0

Hence X is metrizable, by Urysohn's metrizable thm.

Conversely, Suppose X is metrizable $\Rightarrow \exists$ a metric d on X s.t. $\mathcal{T}_d = \mathcal{T}$

$\Rightarrow$ we can regard X as a metric space w.r.t. to metric d.

Since every metric space is normal. Hence X is normal

$\Rightarrow X$ is regular

For a fixed m, we consider $\mathcal{B}_m = \{ B(x, 1/m) \mid x \in X \}$

clearly $\mathcal{B}_m$ is an open covers X.

i.e. $B(x, \frac{1}{m}) = \{ y \in X \mid d(y, x) < 1/m \}$

By metrizability of X, $\mathcal{B}_m$ admits an open refinement $\mathcal{D}$ which covers X and is countably locally finite. such that diameter of each member $\mathcal{B}_m$ is less than or equal to $2/m$

$D \in \mathcal{D} \Rightarrow \exists$ a $B \in \mathcal{B}_m$ s.t. $B \supset D$ $\quad \{\text{s.t. } \operatorname{dm} \mathcal{B}_m \le \frac{2}{m}\}$

$\Rightarrow$ ~~diam~~ $\operatorname{diam} \le \frac{2}{m}$

$\mathcal{B} = \bigcup_{m \in \mathbb{Z}_+} \mathcal{D}$, $\mathcal{D}$ is a refinement of $\mathcal{B}_m$

claim - $\mathcal{B}$ is a basis for X.

By def^n of basis "for $x \in X$ and an open set U containing $x \Rightarrow \exists$ a $B \in \mathcal{B}_m$ s.t. $x \in B \subset U$"

Given $x \in X$ and $\varepsilon > 0$, ~~$x \in CB$~~ $x \in B(x, \varepsilon)$

we choose $m \in \mathbb{Z}_+$ s.t. $\frac{1}{m} < \frac{\varepsilon}{2}$

" every open set in a metric space can be expressed as union of open balls i.e. $x \in U = \bigcup B$" .

Obviously, $\exists$ a $B \in \mathcal{B}_m$ s.t. $x \in B = B(x, 1/m) = B(x, \varepsilon)$

Hence $\mathcal{B}$ is a basis.

Hence X is regular and has a basis that is countably

Imp The following subspace R is compact
$X = \{0\} \cup \{1/n : n \in Z_+\}$

Compactness :- (I) A space X is said to be compact, if every open cover of X possess finite open subcover.

(II) A space X is said to be compact if every open cover of X has a finite open refinement which covers X.

Exq No. 364

Proof (I) ⇒ (II). obvious Every collection is a refinement of its own.

(II) ⇒ (I) Suppose $\mathcal{B}$ is a refinement of X, one can choose for each element of $\mathcal{B}$ an element of $\mathcal{A}$ containing it. In this way one obtaing a finite subcollection of $\mathcal{A}$ that covers X.

Paracompact - A space X is paracompact if every open cover of X has a locally finite open refinement $\mathcal{B}$ that covers X.

Exq - Imp The space R^n is paracompact but it is not compact.

Sol^n - Let $X = R^n$, let $\mathcal{A}$ be an open cover of X
let $B_0 = \phi$ and for each $m \in Z_+$, let B_m is an open ball radius m and centred at the origin.
Given m, choose finitely many element of $\mathcal{A}$ that covers $\overline{B_m}$ and intersects each one with the open set $X - \overline{B_{m-1}}$

Let $\mathcal{C} = \cup \mathcal{C}_m$ is an refinement of $\mathcal{A}$, where $\mathcal{C}_m$ be the collection of open set.
clearly $\mathcal{C}$ is locally finite

For the open set B_m intersects only finitely many element of $\mathcal{C}$.
⇒ $\mathcal{C}$ covers X
⇒ X is paracompact.

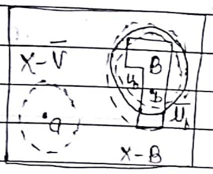
Theorem (41): Every paracompact Hausdorff space X is normal.

Property of paracompact space -

1. A paracompact subspace of a paracompact space is not necessarily paracompact.
2. A closed subspace of a paracompact space is paracompact; Hence $[0,1] \subset R$ with topology, then $[0,1]$ is paracompact.
3. A paracompact Hausdorff space is normal.
4. The product of two paracompact space need not be paracompact.

Thm (41.1). Every paracompact Hausdorff space X is normal.

Proof – ~~First~~



Suppose X is a paracompact and Hausdorff.

(I) let $a \in X - \overline{U_b}$, $b \in U_b$ s.t. $(X - \overline{U_b}) \cap U_b \subset (X - \overline{U_b}) \cap \overline{U_b} = \phi$

$$\Rightarrow (X - \overline{U_b}) \cap U_b = \phi$$

let $\mathcal{A} = \{ U_b \mid b \in B \text{ s.t. } \overline{U_b} \cap a = \phi \} \cup (X - B)$, $\mathcal{A}$ be an open cover of X

By paracompactness of X, $\mathcal{A}$ has an open refinement, say, $\mathcal{C}$, which covers X and is locally finite.

From the subcollection $\mathcal{B}$ of $\mathcal{C}$ consisting of every element of $\mathcal{C}$ that intersects B. Then $\mathcal{B}$ is an open cover of B and $\mathcal{B}$ is locally finite.

If $D \in \mathcal{B}$, then $\overline{D}$ is a disjoint from a. for D intersects B, so $D \subset U_b$, for some $U_b \in \mathcal{A}$.

let $V = \bigcup_{D \in \mathcal{B}} D$, $\exists$

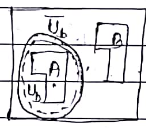
Then V is an open set containing B, because $\mathcal{B}$ is locally finite.

$$\overline{V} = \bigcup_{D \in \mathcal{B}} \overline{D}$$

so that $\overline{V}$ is disjoint from a.

Hence X is regular.

(II) For a fixed $b \in B$, $\exists$ an open set U_b containing A s.t.



$\overline{U_b} \cap b = \phi$

$\Rightarrow U = \bigcap_{b \in B} U_b \supset A$ s.t. $\overline{U} \cap B = \phi$ i.e. $B \subset X - U = V_b$

Thus $A \subset U_b$, $B \subset V_b$ and $U_b \cap V_b = \phi$

~~so X is regular.~~

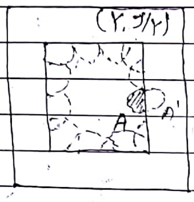
∴ X is normal.

R^n, $R \to$ paracompact, but it is not compact.
$n=1$

Imp

Theorem 41.2 Every closed subspace of a paracompact space is paracompact.

Proof - Let $\mathcal{A}$ be an open cover of Y (w.r.to relative topology on Y) and let Y be a closed subspace of the paracompact space X.



For each $A \in \mathcal{A}$, choose an open set A' of X such that $A' \cap Y = A$

The collection $\mathcal{C} = \{A' \mid A \in \mathcal{A}$ s.t. $A' \cap Y = A\} \cup \{X - Y\}$ be an open cover of X, but X is a paracompact.

$\Rightarrow \exists$ an open refinement $\mathcal{D}$ of $\mathcal{C}$ which covers of X and is locally finite.

Let $\mathcal{B} = \{D \cap Y \mid D \in \mathcal{D}\} \subset \mathcal{A}$, $\mathcal{B}$ an open refinement of $\mathcal{A}$ which covers Y and is locally finite.

$\Rightarrow$ Y is a paracompact.

(I) $(0,1) \to$ paracompact, ~~but~~ (Homeomorphic) to R

(II) $\bar{S}_\Omega \times \bar{S}_\Omega \to$ compact + paracompact

Imp

Lemma 41.3 Let X be regular, and there is an open refinement of X. Then the following cond^n are equivalent.

(1) An open refinement which covers X and is countably locally finite.

(2) A cover of X which is locally finite.

(3) An open cover of X which is locally finite.

(4) A closed cover of X which is locally finite.

Imp

Theorem - Every metrizable space is paracompact.

Proof - Suppose X is metrizable $\Rightarrow X$ is regular,
By lemma 39.2 suppose $\mathcal{A}$ is an open cover of X, it has an open refinement that covers X and is countably locally finite

$\Rightarrow$ $\mathcal{A}$ has an open refinement that covers and is locally finite

$\Rightarrow$ X is paracompact.

Theorem 41.5 Every regular Lindelöf space is paracompact.

Proof - Suppose X is regular space. Let $\mathcal{A}$ be an open cover of X.
Also X is Lindelöff. Then $\exists$ an open refinement ~~of itself~~ $\mathcal{B}$ of $\mathcal{A}$ covering X and countable.
Moreover $\mathcal{B}$ is also locally finite.
$\mathcal{A}$ is refinement of itself.

Since X is regular $\exists$ an open refinement $\mathcal{C}$ of $\mathcal{B}$ i.e. in ~~tern~~ turn $\mathcal{C}$ is open refinement of $\mathcal{A}$ which covers X and is locally finite.

$\Rightarrow$ X is paracompact.

proved.

Ex - The product of two paracompact space need not be paracompact.

The space $\mathbb{R}_l$ is paracompact, for because it is regular and Lindelöff. but $\mathbb{R}_l \times \mathbb{R}_l$ is not paracompact because it is Hausdorff but not normal.

Locally metrizable

The Smirnov Metrization Theorem

Defn - A space X is locally metrizable if every point x of X has a nbd U that is metrizable in the subspace topology.

Imp Theorem (Smirnov Metrization theorem) - A space X is metrizable if and only if it is paracompact and locally metrizable.

Imp proof

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From book 292 (H.K.)

Proof - Suppose X is metrizable. Then X is locally metrizable

$\Rightarrow$ X is paracompact (Stone's theorem - Every metrizable space is paracompact)

Conversely, Suppose that X is paracompact and locally metrizable.
~~We shall show that X has a basis that is countably locally finite.~~
~~Since X is regular~~

Then we have to show that X is metrizable.
~~So proof of this~~
For this, By Nagata-Smirnov metrization theorem ~~i.e.~~

A space X is metrizable if and only if

X is regular and has a basis that is countably locally finite.

Since X is paracompact and paracompact space is Hausdorff; then X is normal space.

X is normal space $\Rightarrow$ X is regular space.

Now Let $A_m = \{ B_c(x, 1/m) : x \in C$ and $c \in \mathcal{C} \}$ be an open cover of X.

$\because$ A_m is paracompact and A_m is an open cover of X. then it has an open refinement $\mathcal{D}_m$ of A_m that covers X.

Let $\mathcal{D} = \bigcup_{m \in \mathbb{Z}^+} \mathcal{D}_m$

then $\mathcal{D}$ is an open cover of X and countably locally finite.

We asserts that $\mathcal{D}$ is a base for X.

So let $x \in X$ and let U_x be a nbd of x

From book 292

A seq^n is a function whose domain is set of natural numbers. $S: N \to X_{(R)}$. $S(1) = S_1, S(2) = S_2$

Net is a generalization of seq^n in which domain is directed set $(\mathcal{D}, \geq)$ ($\geq \to$ follows)

\# Directed set :- "A binary relation $\geq$ on a non-empty set A is said to be directed iff the following axioms are satisfied"

OR

A directed set is a pair $(\mathcal{D}, \geq)$ where D is a non-empty set and $\geq$ a binary relation on D satisfying:

[D_1] For each $a \in \mathcal{D} \Rightarrow a \geq a$

[D_2] For $a, b, c \in \mathcal{D}$, $a \geq b$ and $b \geq c \Rightarrow a \geq c$

[D_3] For $a, b \in \mathcal{D}$ $\exists$ $p \in D$ s.t. $p \geq a$ and $p \geq b$.

Exa - (1) $(N, \geq)$ is a directed set

[D_1] For each $n \in N \Rightarrow n \geq n$

[D_2] For $n, m, p \in N$, $n \geq m$, $m \geq p \Rightarrow n \geq p$

[D_3] For $n, m \in N$, $\exists$ $p (= max(n, m)) \in N$ s.t. $p \geq n$ and $p \geq m$

(3) $(\mathcal{P}(X), \subseteq)$ is a directed set,

[D_1] for each $A \in \mathcal{P}(X) \Rightarrow A \subseteq A$

[D_2] For $A, B, C \in \mathcal{P}(X)$, $A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$

[D_3] For $A, B \in \mathcal{P}(X)$, $\exists\ C \in \mathcal{P}(X)$ s.t. $C \subseteq A$ & $C \subseteq B$.

(2) Let (X, τ) be a topological space and $x \in X$. Also η_x denotes nbd system and define a binary operation '$\geq$' on η_x by $N \geq M$ iff $N \subset M$ $\forall\ N, M \in \eta_x$

Sol^n - [D_1] For each $N \in \eta_x$, $N \subseteq N \Rightarrow N \geq N$

[D_2] For $N, M, L \in \eta_x$, $N \geq M$, $M \geq L \Rightarrow N \subseteq M$, $M \subseteq L$

$\Rightarrow N \subseteq L$ {by transitivity of set inclusion}

$\Rightarrow N \geq L$

[D_3] For $N, M \in \eta_x$, $\exists\ L (= N \cap M) \in \eta_x$ s.t.

$L \subseteq N$, $L \subseteq M \Rightarrow L \geq N$, $L \geq M$

Hence $(\eta_x, \geq)$ is a directed set.

(4) Let (X, τ) be a topological space and $x \in X$, S denotes the set of all subsets of X containing x. define a binary operation '$\geq$' on S by $A \geq B$ iff $A \subseteq B$ $\forall\ A, B \in S$

Whether $(S, \geq)$ is a directed set.

Sol^n - [D_1] For each $A \in S$, $A \subseteq A \Rightarrow A \geq A$

[D_2] For $A, B, C \in S$, $A \geq B$, $B \geq C \Rightarrow A \subseteq B$, $B \subseteq C$

$\Rightarrow A \subseteq C$

$\Rightarrow A \geq C$

[D_3] For $A, B \in S$, $\exists\ P (= A \cap B) \in S$ s.t.

$A \cap B \geq A$, $A \cap B \geq B$

$\Rightarrow P \geq A$, $P \geq B$

Hence $(S, \geq)$ is a directed set.

#

Net - Let $(\mathcal{D}, \geq)$ be a directed set, then any function $S: \mathcal{D} \longrightarrow X$ is said to be a net on X.

exa (1) Let $f: [0,1] \longrightarrow \mathbb{R}$ is bounded and

$\mathcal{D}$ = a partition set of $[0,1] = \{a_0, a_1, a_2, \ldots a_{n-1}, a_n\}$

s.t. $0 = a_0 < a_1 < a_2 < a_{i-1} < a_i < a_{n-1} < a_n = 1$

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Then i^{th} subinterval $= [a_{j-1}, a_j]$
and Let Q = another partition set of $[0,1]$
$= \{b_0, b_1, b_2, \ldots, b_{m-1}, b_m\}$
s.t. $0 = b_0 < b_1 < \ldots < b_{j-1} < b_j < \ldots < b_{m-1} < b_m = 1$

i^{th} subinterval $= [b_{j-1}, b_j]$

P is said to be refinement of Q if each subinterval of partition P is contained in some subinterval of partition Q.

choosen $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$
$\eta = (\eta_1, \eta_2, \ldots, \eta_m)$

Define $D = \{(P, \xi) \mid P \in P[0,1]\}$ ~~is a dire~~ by

$(P, \xi) \geq (Q, \eta)$ iff P is refinement of Q.

then $(D, \geq)$ is a directed set.

Riemann sum $S(P, f) = \sum_{i=1}^{n} f(\xi_i)(a_i - a_{i-1})$

$|(a_{i-1}, a_i)| = a_i - a_{i-1}$

$\lim_{\Delta \to 0} S(P, f) = \int_0^1 f\,dx$ (Rieman integral)

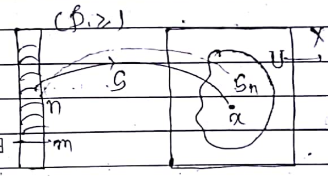
$\Rightarrow S : D \longrightarrow R$

Then S is a net which has image as ~~set~~ Riemann sum.

Limit → sequence
Limit point → set

N → countable
D → arbitrary / countable

Converges of a net / Limit -



(given)

Suppose $S : D \to X$ is a net defined on X and $x \in X$. Then S is said to converges at x if for any given open set U of x, $\exists m \in D$ such that $n \in D$, $n \geq m \Rightarrow S_n \in U$.
The point x is called limit of the ~~set~~ net S.

Theorem - imp Suppose $S : D \to X$ is a net. Then X is Hausdorff if and only if every net on X has unique limit.

Proof - Suppose $S : D \to X$ is a net defined on X and X is Hausdorff. Suppose S converges to distinct point x & y in X.
Then there exist two disjoint open sets U and V s.t. $x \in U$, $y \in V$ and $U \cap V = \phi$.

Case I, When S converges to x, then for open nbd U of x, $\exists m_1 \in D$ s.t. $n \in D$, $n \geq m_1$
$\Rightarrow S_n \in U \longrightarrow (1)$

Case II When S converges to y, then for open nbd of V of y, $\exists\ m_2 \in D$ s.t. $n \in D,\ n \geq m_2 \Rightarrow S_n \in V$. $\longrightarrow (2)$

Now $m_1, m_2 \in D$. then by existence property, $\exists\ m \in D$ s.t. $m \geq m_1,\ m \geq m_2$

Thus $\exists\ m \in D$ s.t. $n \in D,\ n \geq m \Rightarrow S_n \in U$ as well as $S_n \in V$...

$\Rightarrow U \cap V \neq \phi$. a contradiction to $(*)$

Hence our supposition is false, Thus every net in X has unique limit.

Conversely, Suppose On the contrary that X is not Hausdorff.

$\exists$ a pair of distinct points $x, y \in X$ s.t. for any $U \in \eta_x,\ V \in \eta_y$ and $U \cap V \neq \phi$

Take $D = \eta_x \times \eta_y = \{(U,V) \mid U \in \eta_x,\ V \in \eta_y\}$ define

$(U_1,V_1) \geq (U_2,V_2)$ iff $U_1 \subset U_2$ & $V_1 \subset V_2$.

Claim - D is a directed set.

[D1] For each $(U,V) \in D$.

$(U_1,V_1) \subset (U_1,V_1) \Rightarrow (U_1,V_1) \geq (U_1,V_1)$.

[D2] For $(U_1,V_1), (U_2,V_2), (U_3,V_3) \in D$ with

$(U_1,V_1) \geq (U_2,V_2),\ (U_2,V_2) \geq (U_3,V_3) \Rightarrow (U_1,V_1) \subset (U_2,V_2), (U_2,V_2) \subset (U_3,V_3)$

$\Rightarrow (U_1,V_1) \subset (U_3,V_3)$

$\Rightarrow (U_1,V_1) \geq (U_3,V_3)$.

[D3] For $(U_1,V_1), (U_2,V_2) \in D$. then $\exists (U,V) = U_1 \cap U_2,\ V_1 \cap V_2 \in D$ s.t.

$(U,V) \geq (U_1,V_1)$ and $(U,V) \geq (U_2,V_2)$

Hence $(D, \geq)$ is a directed set.

Define $S: D \to X$ by $S(U,V)$ to be any point in $U \cap V$ (By axiom of choice).

Claim - S converges to x

For let G be an open nbd of x. then $(G,X) \in D$ if $(U,V) \geq (G,X)$ in D then $U \subset G$ and so

$S(U,V) \in U \cap V \subset U \subset G$

$\Rightarrow S$ converges to x

Claim - S converges to y

For let H be an open nbd of y, then $(H,X) \in D$. if $(U,V) \geq (H,X)$ in D. then ~~$H \subset$~~ $V \subset H$ and so

$S(U,V) \in U \cap V \subset V \subset H$

$\Rightarrow S$ converges to y

Then we get a ~~condiction~~ contradicting the hypothesis.

Hence X is Hausdorff. proved.

Eventual Subset - Suppose $(D, \geq)$ is the given directed set. Suppose $E \subset D$ then E is said to be eventual subset of D if $\exists\, m \in D$ s.t. $n \in D$, $n \geq m \Rightarrow n \in E$.

A net $S: D \to X$ is said to be eventually in a subset A of X if the set $S^{-1}(A)$ is an eventual subset of D.

Cofinal subset - Let $(D, \geq)$ be a directed set. A subset F of D is said to be cofinal if for each $m \in D$, $\exists\, n \in D$ s.t. $n \geq m$ and $n \in F$.

A net $S: D \to X$ is said to be frequently in a subset A of X if $S^{-1}(A)$ is a cofinal subset of D.

Exa - Let $D = N$, '$\geq$' then $(N, \geq)$ is a directed set

$E = \{5, 6, 7, \ldots\} \subset N$

Q. State whether E is eventual or cofinal subset of N.

Solve - $\exists\, m (=5) \in N$ s.t. $n \in N$, $n \geq m \Rightarrow n \in E$.

So E is eventual subset of N

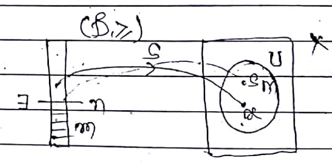
Q. $F = \{1, 3, 5, 7, \ldots\} \subset N$

Solve - For each $m (=1) \in N$, $\exists\, n \in N$ s.t. $n \geq m$ and $n \in F$

$\Rightarrow$ F is cofinal subset of N.

Every eventual subset is a cofinal subset but converse need not be true.

Cluster point -



[Suppose $(D, \geq)$ is a directed set and let $x \in X$ then x is said to be cluster point. If for any open set U of x, $\exists\, n \in D$ s.t. $m \in D$ $m \geq n \Rightarrow S_m \in U$.] X

Imp

Let $S: D \to X$ be a net. A point $x \in X$ is said to be a cluster point of S if for ~~any~~ every nbd U of x in X and $m \in D$, $\exists\, n \in D$ s.t. $n \geq m$ and $S_n \in U$.

Proposition – If $(\mathcal{D}, \geq)$ is a directed set and E is an eventual subset of $\mathcal{D}$ $(E \subset \mathcal{D})$ then E, with the restriction of $\geq$ is a directed set. $(\geq' = \geq|_E)$

Proof – Let $\geq'$ denote the relation $\geq$ restricted to E.
Then we have to show that $(E, \geq')$ is a directed set.
For this

[1] For each $a \in E$, $E \subset \mathcal{D}$ and $\mathcal{D}$ is a directed

$\Rightarrow a \in \mathcal{D}$ and $a \geq a$ but $a \in E$

$\Rightarrow a \geq' a$.

[2] For $a, b, c \in E$, $a \geq' b$, $b \geq' c$ and $E \subset \mathcal{D}$

$\Rightarrow a, b, c \in \mathcal{D}$, $a \geq b$, $b \geq c$

$\Rightarrow a, b, c \in \mathcal{D}$, $a \geq c$ but $a, b, c \in E$

$\Rightarrow a \geq' c$

[3] For $a, b \in E$, $E \subset \mathcal{D} \Rightarrow a, b \in \mathcal{D}$, but $\mathcal{D}$ is a directed set by $\geq$

$\Rightarrow \exists\, p \in \mathcal{D}$ s.t. $p \geq a$ & $p \geq b$

Since E is eventual subset of $\mathcal{D}$,
$\exists\, m \in \mathcal{D}$ s.t. $n \in \mathcal{D}$, $n \geq m \Rightarrow n \in E$.

So $p \geq a$, $a \geq m$ because $a \in E$

$\Rightarrow p \geq m$, $p \in \mathcal{D}$

$\Rightarrow p \in E$

Again $a, b \in E$ and $p \in E$ s.t. $p \geq a$ & $p \geq b$

$\Rightarrow \exists\, p \in E$ s.t. $p \geq' a$ and $p \geq' b$

Thus, $(E, \geq')$ is a directed set.

Theorem – Let $S : \mathcal{D} \to X$ be a net defined on X. Let F be a cofinal subset of X. If $S|_F : F \to X$ converges to x, then show that S clusters at x.

Proof – Suppose $S|_F : F \to X$ converges to $x \in X$.
Let U be any open set containing x.
then $\exists\, m_1 \in F$ s.t. $n \in F$, $n \geq m_1 \Rightarrow S_n \in U \longrightarrow (1)$

Let m be arbitrary in $\mathcal{D}$, but fixed, because F is cofinal subset of $\mathcal{D}$, then $\exists\, m \in F$ s.t. $m_2 \geq m$

Now $m_1, m_2 \in F \Rightarrow \exists\, m' \in F$ s.t. $m' \geq m_1$ & $m' \geq m_2$

$\Rightarrow \exists\, m' \in \mathcal{D}$ and $m' \geq m_1'$ s.t. $S_{m'} \in U$. {by (1)}

$\Rightarrow \exists\, m \in \mathcal{D}$ s.t. $m' \geq m_2$, $m_2 \geq m$ and $S_{m'} \in U$

$\Rightarrow \exists\, m \in \mathcal{D}$ s.t. $m' \geq m$ and $S_{m'} \in U$

In particular, $S_m \in U$, but U and m are arbitrary

$\Rightarrow S$ clusters at α.

Problem - Let $\{x\} \subset \mathbb{R}$, then $x \in \mathbb{R}$ is the limit point of $\{x_n\}$ iff $\exists$ a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to α.

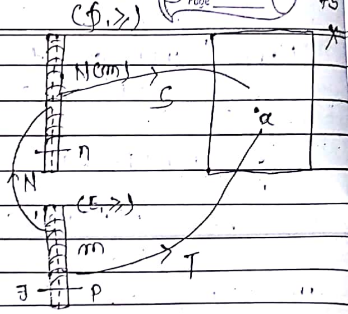
Exa - Let $\{\frac{1}{n}\} \subset \mathbb{R}$, then $0 \in \mathbb{R}$ is the limit point of $\{\frac{1}{n}\}$ iff $\exists$ a subsequence $\{n_i = 2i+1\}$

$$\{x_{n_i}\} = \{x_3, x_5, x_7, \dots\} = \{\tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \dots\}$$

s.t. $\lim\limits_{i \to \infty} x_{n_i} = \lim\limits_{i \to \infty} \frac{1}{n_i} = \lim\limits_{i \to \infty} \frac{1}{2i+1} = 0.$

Subnet -

Let $S: \mathcal{D} \to X$ and $T: E \to X$ be nets. Then T is said to be subnet of S if there exist a fun^n $N: E \to \mathcal{D}$ such that



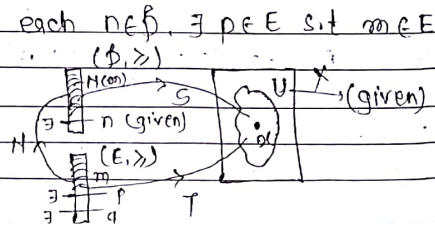
(i) $S \circ N = T$
(ii) For each $n \in \mathcal{D}$, $\exists\, p \in E$ s.t. $m \in E$, $m \geq p \Rightarrow N(m) \geq n$

Theorem - Let $S: \mathcal{D} \to X$ be a net defined on ~~and~~ then S clusters at $\alpha \in X$ iff every subnet of S converges to α.

Proof ($\Leftarrow$) Suppose $T: E \to X$ is a subnet of S which converging to $\alpha \in X$.

Then $\exists$ a function $N: E \to \mathcal{D}$ s.t.

(i) $S \circ N = T$
(ii) For each $n \in \mathcal{D}$, $\exists\, p \in E$ s.t. $m \in E$, $m \geq p \Rightarrow N(m) \geq n$



Now T converges to x $\Rightarrow \exists\, q \in E$ s.t. $m_1 \in E$, $m_1 \geq q$ and $T(m_1) \in U$

$\Rightarrow \exists\, q \in E$ s.t. $m_1 \in E$, $m_1 \geq q$ and $(S \circ N)(m_1) \in U$

i.e. $S(N(m_1)) \in U$

Now $p, q \in E \Rightarrow \exists\, m \in E$ s.t. $m \geq p$, $m \geq q$ and so $S(N(m)) \in U$, also $N(m) \geq n$

$\Rightarrow \exists\, N(m) \in D$ s.t. $N(m) \geq n$ and $S(N(m)) \in U$

$\Rightarrow S$ cluster at $x \in X$, because U and n are arbitrary.

Conversely, ($\Rightarrow$) Suppose $S : D \to X$ cluster at $x \in X$.

We construct a subnet T of S as follows:
Let η_x be the nbd system of the point $x \in X$.
Let $\geq$ denote the given binary relation on the directed set D. Then we define

$$\left(E = \{(n, U) \in D \times \eta_x \mid S_n \in U\}, \geq\right)$$

For $(n, U), (m, V) \in E$, let $(n, U) \geq (m, V)$ iff $n \geq m$ & $U \subset V$

Claim 1. "$\geq$" directs E, so that $(E, \geq)$ becomes a directed sets.

[D1] For each $(n, U) \in E$, $(n, U) \subset (n, U)$ $\{n \geq n$ & $U \subset U\}$

$\Rightarrow (n, U) \geq (n, U)$

[D2] For $(n, U), (m, V), (p, W) \in E$ with

$(n, U) \geq (m, V)$, $(m, V) \geq (p, W) \Rightarrow n \geq m$, $m \geq p$ & $U \subset V$, $V \subset W$

$\Rightarrow n \geq p$ and $U \subset W$

$\Rightarrow (n, U) \geq (p, W)$

[D3] For $(n, U), (m, V) \in E$, then $n, m \in D$, $U, V \in \eta_x$

$\Rightarrow \exists\, p \in D$ s.t. $p \geq n$, $p \geq m$ and that

$U \cap V \in \eta_x \longrightarrow$ (1)

Set p as defined in (1) and $W = U \cap V \in \eta_x$
Then we see that $\exists\, (p, W) \in E$ s.t.
$(p, W) \geq (n, U)$ and $(p, W) \geq (m, V)$

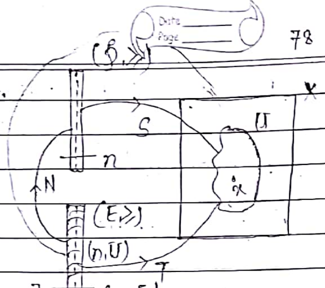
Thus, $(E, \geq)$ is a directed set.

Now define $T : E \to X$ by $T(n, U) = S_n$ for $(n, U) \in E$

Claim 2. T is indeed a subnet of S

i.e. Define $N : E \to D$ by $N(n, U) = n$

then $(S \circ N) : E \longrightarrow X$ ~~by~~ s.t.



(i)

$$(S \circ N)(n, U) = S(N(n, U))$$
$$= S(n) = S_n$$
$$= T(n, U)$$

$\Rightarrow \quad S \circ N = T$

(ii)

"It only remains to verify that T converges to x.
For this,

Let G be a nbd of x and let $m \in D$ such that $S_m \in G$. Then $(m, G) \in E$, because S clusters at x, it follows that for any nbd G of x and any $n \in D$, $\exists\, m \in G$ s.t. $m \geq n$ and $S_m \in G$.

Now $(n, U) \in E$, $(n, U) \geq (m, G) \Rightarrow T(n, U) = S_n \in U \subset$

$\Rightarrow \quad n \geq m, \; U \subset G$

$\Rightarrow \quad T$ converges to x.

proved.

Theorem - Let $A \subset X$, $x \in \bar{A}$ iff $S : D \longrightarrow A$ converges to x, when it is regarded as a net defined on X.

Proof - $(\Leftarrow)$ Let U be any open nbd of x (in X). Since S converges to x, it follows that $\exists\, m \in D$ such that $\forall\, n \in D$, $n \geq m$

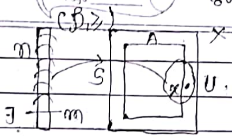
$\Rightarrow S_n \in U$, but by definition $S : D \rightarrow A$, $S_n \in A$

$\Rightarrow \quad S_n \in U \cap A$

$\Rightarrow \quad U \cap A \neq \phi$

$\overline{A} = \{x \in X \mid U \in \eta_x, U \cap A \neq \phi\}$

$\Rightarrow x \in \overline{A}$



Conversely. $(\Rightarrow)$ ~~Let $x \in \overline{A} \Rightarrow$ for each $U \in \eta_x, U \cap A \neq \phi$~~ suppose $x \in \overline{A}$; then every nbd U of x intersects A non-vocausly i.e. $U \cap V \neq \phi \quad \forall U \in \eta_x$.

Let η_x be the nbd of $x \in X$

Claim $(\eta_x, \geq)$ directed as usual.

Define $S : \eta_x \longrightarrow A$ by $S(U) = x_U$ = any point in $U \cap A$ {by axiom of choice}

Now we intend to show that S converges to $x \in X$.

Let G be any open set containing x. Then for any $U \in \eta_x$ s.t. $U \geq G$ implies $U \subseteq G$ and $S(U) = x_U \in U \subseteq G$

$\Rightarrow S$ converges to $x \in X$.

proved.

Corollary - A subset A of a space X is closed iff limits of nets in A are in A.

Proof - Proof as same previous theorem, A set is closed iff $A = \overline{A}$

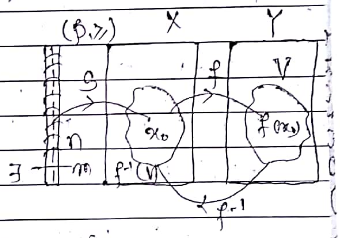
Theorem. Let $B \subset X$, then B is open iff no net in $X - B$ can converges to a point in B.

Proof. - Apply the previous corollary to $X - B$.

Theorem - Imp. Let $f : X \to Y$ be a function and let $x_0 \in X$. Then f is continuous at x_0 iff whenever a net $S : \mathcal{D}' \to X$ converges to x_0 then the net $f \circ S : \mathcal{D} \to Y$ converges to $f(x_0)$.

Proof :- $(\Rightarrow)$ Suppose $f : X \to Y$ is continuous at x_0.

Let V be any open nbd of $f(x_0)$ in Y. Then $f^{-1}(V)$ is an open nbd of x_0 in X.



Suppose there is a net $S : \mathcal{D} \to X$ which converges to x_0. Then for open nbd $f^{-1}(V)$, $\exists\, m \in \mathcal{D}$ s.t. $\forall n \in \mathcal{D}, n \geq m$. $S(n) \in f^{-1}(V)$

$\Rightarrow f(S(n)) = (f \circ S)(n) \in f[f^{-1}(V)] \subset V$.

Thus, for open nbd V of $f(x_0)$, $\exists\, m \in \mathcal{D}$ s.t. $n \in \mathcal{D}, n \geq m \Rightarrow (f \circ S)(n) \in V$

Hence, the net $f \circ S : \mathcal{D} \to Y$ converges to $f(x_0)$ in Y.

Conversely ($\Leftarrow$), Suppose the given condition holds good. Then we have to show that f is continuous at x_0.

If not, then $\exists$ an open nbd V of $f(x_0)$ s.t. $f^{-1}(V)$ is not an open nbd of x_0.

$\Rightarrow$ $f^{-1}(V)$ does not continuous any nbd of x_0 in X

$\Rightarrow$ $\forall N \in \eta_{x_0}$, $N \cap (X - f^{-1}(V)) \neq \phi$.

Let $(\eta_{x_0}, \geq)$; $U \geq V \Leftrightarrow U \subset V$ is a directed set.

(I) Define $S : \eta_{x_0} \to X$ by

$S(N) = x_N$ = any point in $N \cap (X - f^{-1}(V))$.

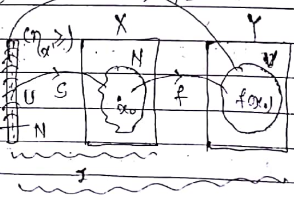
Then S converges to x_0 in X. For any given nbd N of x_0, $\exists N \in \eta_{x_0}$ s.t. $\forall U \in \eta_{x_0}$, $U \geq N$

$\Rightarrow$ $U \subset N$ and $S(U) = x_U$ = a point in $U \cap (X - f^{-1}(V))$

$\Rightarrow$ $S(U) \in U \subset N$

This confirms that S converges to $x_0 \in X$.

(II) For any $N \in \eta_{x_0}$, we have



$(f \circ S)(N) = f(S(N)) = f$ (a point in $N \cap (X - f^{-1}(V))$) $\in Y - f^{-1}(V)$

$\Rightarrow$ $(f \circ S) N \in f(X - f^{-1}(V)) = Y - V$

$\Rightarrow$ $f \circ S$ will never converges to $f(x_0)$, contradicts the hypothesis.

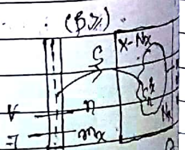
So f is continuous at x_0. proved

Theorem For a topological space X, the following statements are equivalent.

1. X is compact
2. Every net ~~on~~ in X has a cluster point in X
3. ~~Every net in X has~~
3. $\exists$ a subnet of S which converges to the point x of X.

Proof :- (i) $\Rightarrow$ (ii) Suppose X is compact. Let $S : D \to X$ be a net defined on X. Suppose S has no cluster point in X.

Then for each $x \in X$, $\exists$ an open nbd N_x and an element $m_x \in D$ s.t.



$\forall n \in D$, $n \geq m_x \Rightarrow S_n \notin N_x$, i.e. $S_n \in X - N_x$

On the other hand, the family $\{N_x : x \in X\}$ is an open cover of X. i.e.

$X = \bigcup_{x \in X} N_i$, Because X is compact,

By compactness, $\exists\ x_i\ (i = 1, 2, \ldots n)$ s.t. $X = \bigcup_{i=1}^{n} N_{x_i}$ ⟶ (2)

let $m_{x_1}, m_{x_2}, \ldots m_{x_n} \in \mathcal{D} \Rightarrow \exists\ m \in \mathcal{D}$ s.t. $m \geq m_i$, because $\mathcal{D}$ is a directed set.

Now rewriting (1) as: for all x_i's $(i = 1, 2, \ldots n)$
$\exists\ m \in \mathcal{D}$ s.t. $n \in \mathcal{D}$, $n \geq m \Rightarrow S_n \in \bigcap_{i=1}^{n} (X - N_{x_i})$

$\Rightarrow S_n \in X - \bigcup_{i=1}^{n} N_{x_i} = X - X = \phi$, a contradiction.

$\Rightarrow \exists$ a point x in X, that is, cluster point of S.

(ii) ⇒ (i) $\boxed{X \text{ is compact} \Leftrightarrow \text{Every family of closed sets having finite intersection property has non-empty intersection}}$

Suppose $\mathcal{C}$ is the family of closed subset of X having f.i.p., suppose $\mathcal{D}$ is the family of all finite intersection of members of $\mathcal{C}$. clearly, $\mathcal{D} \subset \mathcal{C}$

Claim (i) No member of $\mathcal{D}$ is empty.
(ii) For $D, E \in \mathcal{D}$, we define $D \geq E \Leftrightarrow D \subset E$

* Indian Star * Pancham *

i.e. $(\mathcal{D}, \geq)$ is a directed set. because whenever $D, E \in \mathcal{D}$, $D \cap E \in \mathcal{D}$ and $D \cap E \geq D$, $D \cap E \geq E$.

Define $S : \mathcal{D} \to X$ by $S(D) = x_D$ = any point in D
By hypothesis, S has a cluster point x in X.

~~We~~ claim - $x \in \bigcap_{C \in \mathcal{C}} C$.

If not, then $\exists$ a $C \in \mathcal{C}$ s.t. $x \notin C$, i.e. $x \in X - C$ is a nbd of x. Also $C \in \mathcal{D}$. So by definition of a cluster point, $\exists\ D \in \mathcal{D}$ s.t. $D \geq C$ and $S(D) \in X - C$, but $D \subset \cancel{C} \Rightarrow X - C \subset X - D$

$\Rightarrow S(D) \in X - C \subset X - D$., contradicting that $S(D) \in D$

So $x \in \bigcap_{C \in \mathcal{C}} C \Rightarrow \bigcap_{C \in \mathcal{C}} C \neq \phi$

$\Rightarrow X$ is compact.

proved.

"Filter"

Definition – A Filter on a set X is a nonempty family $\mathcal{F}$ of subsets of X such that

(i) $\phi \notin \mathcal{F}$

(ii) $\mathcal{F}$ is closed w.r.to finite intersections and

(iii) if $B \in \mathcal{F}$ and $A \supset B$ $(B \subset A)$ then $A \in \mathcal{F}$, i.e. $\mathcal{F}$ is closed w.r. to superset condition, $\forall A, B \subset X$.

Exa – ① $\{X\}$ (singleton family) is a filter on X.

② $\phi \neq A \subset X$, then $\mathcal{F} = \{S \subset X \mid S \supset A\}$ is a atomic filter, and A being Atom called the atom of the filter.

③ If X is infinite set, then $\mathcal{F}$ = set of all cofinite subsets, is ~~called~~ a filter on X.

i.e. "$A \in \mathcal{F} \Leftrightarrow X - A$ is finite"

Sol^n/ (i) $\phi \notin \mathcal{F} \Rightarrow X - \phi = X$ i.e. $\phi \notin \mathcal{F}$

(ii) if $A, B \in X \Rightarrow X - A = \{a_1, a_2, \dots, a_m\}$ &
$X - B = \{b_1, b_2, \dots, b_k\}$ both are finite

$\Rightarrow (X-A) \cup (X-B)$ = a finite set

$\Rightarrow X - (A \cap B)$ = —"—

$\Rightarrow A \cap B \in \mathcal{F}$.

(iii) if $B \in \mathcal{F}$, $A \supset B \Rightarrow X - B$ = a finite set

But $A \supset B \Rightarrow X - A \subset X - B$

So that $X - A$ = a finite set

$\Rightarrow A \in \mathcal{F}$!

So $\mathcal{F}$ is a filter on X. It is called a cofinite filter.

\# Let $S : D \to X$ be a net defined on X, for any $m \in D$, $S_m = \{S(n) \in X \mid n \in D, n \geq m\} \subset X$, then $\mathcal{F} = \{A \subset X \mid A \supset S_m, m \in D\}$ is filter generated by net S. \#

make

Imp

Base of a filter :- Let $\mathcal{F}$ be a filter on a set X. Then a sub-family $\mathcal{B}$ of $\mathcal{F}$ is said to be a base for $\mathcal{F}$, if for any $A \in \mathcal{F}$, $\exists$ a $B \in \mathcal{B}$ such that $B \subset A$.

Exa $X = \{1, 2, 3, 4, 5\}$, then $\mathcal{F} = \{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,3,4,5\}, \{1,2,4,5\}, X\}$

$$\left.\begin{array}{l} {}^4C_1 = 4 \\ {}^4C_2 = 6 \\ {}^4C_3 = 4 \\ {}^4C_4 = 1 \end{array}\right\} \text{sets}$$

then $\mathcal{B} = \{\{1\}\}$ is a base for $\mathcal{F}$.

Theorem - Let β be a family of non-empty subsets of a set X. Then $\exists$ a filter $\mathcal{L}$ having β as a base iff β has the property that for any $B_1, B_2 \in \beta$, $\exists$ a $B_3 \in \beta$ such that $B_1 \cap B_2 \supset B_3$.

Proof ($\Rightarrow$) Suppose $\mathcal{L}$ is a filter having β as a base, then $\beta \subset \mathcal{L}$ and $\phi \notin \mathcal{L}$, hence $\phi \notin \beta$.

Let $B_1, B_2 \in \beta$. $\beta \subset \mathcal{L} \Rightarrow B_1, B_2 \in \mathcal{L}$

$\Rightarrow B_1 \cap B_2 \in \mathcal{L}$ and β is a base for $\mathcal{L}$.

$\Rightarrow \exists$ a $B_3 \in \mathcal{L}$, $B_3 \in \beta$ s.t. $B_1 \cap B_2 \supset B_3$.

Conversely ($\Leftarrow$) Suppose given condition holds good.

Define $\mathcal{L} = \{A \subset X \mid A \supset B$ for some $B \in \beta\}$

Claim - $\mathcal{L}$ is a filter on X.

(i) $\phi \notin \mathcal{L}$, because empty set cannot be superset of a non-empty set.

(ii) Let $A_1, A_2 \in \mathcal{L} \Rightarrow A_1 \cap A_2 \in \mathcal{L}$

For this, $A_1, A_2 \in \mathcal{L} \Rightarrow \exists B_1, B_2 \in \beta$ s.t. $A_1 \supset B_1, A_2 \supset B_2$

$\Rightarrow \exists$ a $B_3 \in \beta$ s.t. $A_1 \cap A_2 \supset B_1 \cap B_2 \supset B_3$

$\Rightarrow A_1 \cap A_2 \subset X$ and $A_1 \cap A_2 \supset B_3$

$\Rightarrow A_1 \cap A_2 \in \mathcal{L}$.

(iii) If $C \in \mathcal{L}$, $A \supset C \Rightarrow \exists B \in \beta$ s.t. $C \supset B$ and $A \supset C$

$\Rightarrow \exists B \in \beta$ s.t. $A \supset B$

$\Rightarrow A \in \mathcal{L}$, i.e. $\mathcal{L}$ is closed w.r.to superset condition.

Thus $\mathcal{L}$ is a filter on X and β is a base for it by its very construction.

Corollary: Let β be a family of non-empty subsets of X, then β is a base for a filter if β is closed w.r.to finite intersection.

Proof - Let $B_1, B_2 \in \beta \Rightarrow B_1 \cap B_2 \in \beta$

set $B_3 = B_1 \cap B_2 \Rightarrow \exists$ a $B_3 \in \beta$ s.t.

$B_1 \cap B_2 \in \beta$

$\Rightarrow \beta$ is closed w.r.to finite intersection.

$\mathcal{S}$ is sub-base $\equiv \mathcal{B} = \{ \bigcap_{i=1}^{n} S_i \mid n \in N_f, S_i \in \mathcal{S} \}$ (base)

Subbase :- Let $\mathcal{L}$ be a filter on a set X. Then a subfamily $\mathcal{S}$ of $\mathcal{L}$ is ~~said to be~~ a sub-base for $\mathcal{L}$ such that all finite intersection of members of $\mathcal{S}$ is a base for $\mathcal{L}$.

In such case, we say that $\mathcal{S}$ generates $\mathcal{L}$.

Theorem - Let $\mathcal{S}$ be a family of subsets of a set X. Then $\mathcal{S}$ is a sub-base for a filter $\mathcal{L}$ iff $\mathcal{S}$ has f.i.p.

Proof :- ($\Rightarrow$) Suppose $\mathcal{S}$ is a sub-base for a filter $\mathcal{L}$. then $\mathcal{S} \subset \mathcal{L}$ and that $\mathcal{L}$ has f.i.p.

$\Rightarrow$ $\mathcal{S}$ has f.i.p.

Conversely, ($\Leftarrow$) Suppose $\mathcal{S}$ has f.i.p.

Let $\mathcal{B}$ denotes the family of all finite intersections of members of $\mathcal{S}$.

Define $\mathcal{B} = \{ \bigcap_{i=1}^{n} S_i \mid n \in N_f, S_i \in \mathcal{S} \}$

clearly $\phi \notin \mathcal{B}$ and $\mathcal{B}$ is closed under finite intersection.

But Any family which does not contain the empty set and which is closed under finite intersection is a base for a unique filter.

So $\mathcal{B}$ is a base for $\mathcal{L}$ on X

thus $\mathcal{S}$ is a sub-base for $\mathcal{L}$ (by very definition of $\mathcal{B}$)

Proved.

(Limit)

Converges of a filter - Let $(X, \mathcal{J})$ be a topological space and let $\mathcal{L}$ be a filter on X. A point $x \in X$ is said to be converge at x if $\eta_x \subset \mathcal{L}$ (or $\mathcal{L}$ is said to be a limit w.r.t. $\mathcal{J}$ if every nbd of x belong to $\mathcal{L}$)

Also a point $y \in X$ is said to be a cluster point of $\mathcal{L}$. if every nbd of y intersect every member of $\mathcal{L}$. (or if each $U \in \eta_x$ intersect each $F \in \mathcal{L}$).

Que. If x is a limit of a filter $\mathcal{L}$, then x is also a cluster point of $\mathcal{L}$.

Ans - Beause $\eta_x \subset \mathcal{L}$ and $\mathcal{L}$ has f.i.p.

So every $U \in \eta_x$ intersects each $F \in \mathcal{L}$

i.e. $U \cap F \neq \phi$

Thus x ~~cluster~~ becomes cluster point.

But the converse need not be true.

Q. If $\mathcal{F}$ converges to x and $\mathcal{G} \supset \mathcal{F}$ then a filter $\mathcal{G}$ converges to x or not.?

Ans. ~~yes~~ $\mathcal{G}$ converges to x, because if $\mathcal{F}$ converges to x.

then $\eta_x \subset \mathcal{F}$ and $\mathcal{F} \subset \mathcal{G}$

$\Rightarrow \eta_x \subset \mathcal{G}$

$\Rightarrow \mathcal{G}$ converges to x

$\mathcal{G}$ is a subfilter of $\mathcal{F}$ although it is superset of $\mathcal{F}$ w.r.t. set inclusion.

Q. $\mathcal{F}$ is a cluster filter which clusters at x whether $\mathcal{G}$ converges or cluster at x.? ($\mathcal{G} \supset \mathcal{F}$)

Ans. Since $\mathcal{F}$ cluster at x ~~and $\mathcal{G} \supset \mathcal{F}$~~. then $U \in \eta_x$ intersects each $F \in \mathcal{F}$

Let $S = \eta_x \cup \mathcal{F}$ having f.i.p

$\Rightarrow \exists$ a filter $\mathcal{G} \supset \mathcal{F}$ having S as a subbase

$\Rightarrow \mathcal{G}$ converges to x, because $\eta_x \subset \mathcal{G}$ and hence cluster at x.

Net $\Leftrightarrow$ cluster for generated $\Leftrightarrow$ cluster & converges;

v.D.
Th. imp

\# Let $S : D \to X$ be a net. then S converges to x as net, iff the filter $\mathcal{F}$ generated by S converges to x as filter

Proof ($\Rightarrow$) Assume S converges to x. Let $U \in \eta_x$ be given. then $\exists\ m \in D$ s.t. $n \in D$.

$n \geq m \Rightarrow S(n) \in U$

$\Rightarrow B_m \subset U$ for some $m \in D$

~~But~~ $\Rightarrow U \in \mathcal{F}$, where $\mathcal{F}$ is a filter generated by a net S.

But U was an arbitrary nbd of x.

So $\eta_x \subset \mathcal{F}$.

$\Rightarrow \mathcal{F}$ is a converges to x as a filter.

Conversely. ($\Leftarrow$) Suppose the generated filter $\mathcal{F}$ converges to x as filter.

then $\eta_x \subset \mathcal{F}$ { by definition of converges to filter }

For a choose. $N \in \eta_x$. It is evident that $N \in \mathcal{F}$.

$\Rightarrow \exists\ m \in D$ s.t. $N \supset B_m = \{S(n) \mid n \in D,\ n \geq m\}$

$\Rightarrow \exists\ m \in D$ s.t. $n \in D$, $n \geq m$ and $S(n) \in N$

$\Rightarrow S$ converges to x as net.

proved.

Let $\mathcal{F}$ be a filter in a space X and S be the associated net in X. Let $x \in X$. Then $\mathcal{F}$ converges to x as a filter iff S converges to x as a net.

Proof - (Similarly same proof as previous theorem.)

($\Rightarrow$) Suppose $\mathcal{F}$ converges to x. Let $x \in X$ then $\eta_x \subset \mathcal{F}$ {by definition}

Let U be any arbitrary open nbd of x, then $U \in \eta_x$, but $\eta_x \subset \mathcal{F} \Rightarrow U \in \mathcal{F}$

$\Rightarrow \exists\, m \in \beta$ s.t. $n \in \beta$, $n \geq m \Rightarrow S(n) \in U$

$\Rightarrow S$ converges to x as a net.

($\Leftarrow$) Suppose S converges to x as a net.
Then for an open ~~set~~ nbd U of x,
$\exists\, m \in \beta$ s.t. $n \in \beta$, $n \geq m \Rightarrow S(n) \in U$

$\Rightarrow \exists\, m \in \beta$ s.t. $U \supseteq B_m = \{S(n) \mid n \in \beta, n \geq m\}$

but U was an arbitrary, so

$U \in \eta_x \Rightarrow \eta_x \subset \mathcal{F}$

$\Rightarrow \mathcal{F}$ converges to x as a filter.

proved.

Imp

Theorem - A topological space X is Hausdorff iff every filter on X has unique limit.

Proof - Suppose X is a Hausdorff. Suppose on the contrary, that $\exists$ a filter $\mathcal{F}$ on X which converges to x as well as y.

Since X is Hausdorff. $\exists\, U \in \eta_x$ & $V \in \eta_y$ s.t. $U \cap V = \phi$ $\longrightarrow$ (1)

Case I - When $\mathcal{F}$ converges to x, then $\eta_x \subset \mathcal{F}$
Case II - When $\mathcal{F}$ converges to y, then $\eta_y \subset \mathcal{F}$.

From case (I) & (II), we get

$U, V \in \mathcal{F} \Rightarrow U \cap V \neq \phi$, because $\mathcal{F}$ has f.i.p,

a contradiction to (1)

So every filter on X has unique limit.

Conversely; Suppose every filter on X has unique limit.

If X is not Hausdorff, every $\exists\, x, y \in X$, $x \neq y$ s.t. every nbd of x intersects every nbd of y

$\Rightarrow \eta_x \cup \eta_y$ has the f.i.p.

$\Rightarrow \exists$ a filter $\mathcal{F}$ on X containing $\eta_x \cup \eta_y$

$\Rightarrow \mathcal{F}$ converges both to x and y, a contradicting the hypothesis.

$f(\mathcal{L})$ is not necessarily filter.

$\Rightarrow$ X is a Hausdorff.

Proved.

Theorem :- For a topological space X, the following statements are equivalent.

(1) X is compact.
(2) Every filter on X has a cluster point in X.
(3) Every filter on X has a convergent subfilter.

Let $f: X \to Y$ be a function and $\mathcal{L}$ a filter on X. Then the family $f(\mathcal{L}) = \{f(A) : A \in \mathcal{L}\}$ is a base for a filter on Y.

Proof - Any collection of non-empty subsets of X is a base for a filter $\mathcal{L}$ on X if the collection is closed w.r.t. finite intersection.

Then $f(\mathcal{L})$ is a base for some filter on Y.

(i) $\phi \notin f(\mathcal{L})$, because $\phi \notin \mathcal{L}$ and therefore $f(A) \neq \phi$, for any $A \in \mathcal{L}$.

(ii) $B_1, B_2 \in f(\mathcal{L}) \Rightarrow \exists\ A_1, A_2 \in \mathcal{L}$ s.t. $B_1 = f(A_1)$ & $B_2 = f(A_2)$

$\Rightarrow \exists\ A_1 \cap A_2 \in \mathcal{L}$ s.t. $B_1 \cap B_2 = f(A_1) \cap f(A_2) = f(A_1 \cap A_2)$

$\Rightarrow$ $B_1 \cap B_2 \in f(\mathcal{L})$

proved.

obs.

$f_{\#}(\mathcal{L})$ is a filter for which $f(\mathcal{L})$ is a base.

$f_{\#}(\mathcal{L}) \to$ image filter of $\mathcal{L}$.

Imp.

Let $f: X \to Y$ be a topological space, $x \in X$ and $\mathcal{L}$ is a filter on X. Then f is continuous at x iff whenever $\mathcal{L}$ converges to x, then $f_{\#}(\mathcal{L})$ converges to $f(x)$.

Proof ($\Rightarrow$) Suppose $f: X \to Y$ is continuous at $x \in X$. and suppose $\mathcal{L}$ converges to x.

suppose N is a nbd of $f(x)$, then By continuity of f, $f^{-1}(N)$ is a nbd of x.

Since $\mathcal{L}$ converges to $x \Rightarrow f^{-1}(N) \in \mathcal{L}$.

$\Rightarrow f(f^{-1}(N)) \in f(\mathcal{L}) \subset f_{\#}(\mathcal{L})$

$\Rightarrow f(f^{-1}(N)) \in f_{\#}(\mathcal{L})$, but $N \supset f(f^{-1}(N))$

$\Rightarrow N \in f_{\#}(\mathcal{L})$, but N was arbitrary.

$\Rightarrow \eta_{f(x)} \subset f_{\#}(\mathcal{L})$

This shows that $f_{\#}(\mathcal{L})$ converges to $f(x)$.

($\Leftarrow$) Suppose $f_{\#}(\mathcal{L})$ is converges to $f(x)$. Then we have to show that f is continuous at $x \in X$.

If not, then $\exists$ a nbd N of $f(x)$ in Y s.t. $f^{-1}(N)$ is not nbd of x in X.

$\Rightarrow S \subset \mathcal{L}$

$\Rightarrow \eta_x \subset \mathcal{L}$

$\Rightarrow \mathcal{L}$ converges to x

$\Rightarrow f_{\#}(\mathcal{L})$ converges to $f(x)$

$\Rightarrow N \in f_{\#}(\mathcal{L})$. $\longrightarrow$ (1)

Again, $X - f^{-1}(N) \in \mathcal{L}$

$\Rightarrow f(X - f^{-1}(N)) \in f(\mathcal{L}) \subset f_{\#}(\mathcal{L})$, but $Y - N \supset f(X - f^{-1}(N))$.

$\Rightarrow Y - N \in f_{\#}(\mathcal{L})$ $\longrightarrow$ (2)

From (1) and (2), we conclude that $~~N \cap (X~~ \; N \cap (Y - N) \neq \phi$ by f.i.p.

Thus $f_H(\mathcal{L})$ con does not converges to $f(x)$, a contradiction.

Hence f is a continuous ~~at~~ ~~of~~ at x.

proved.

IMP

Theorem - Let X be the topological product of an indexed family of spaces $\{X_i : i \in I\}$. Let $\mathcal{L}$ be a filter on X and let $x \in X$. Then $\mathcal{L}$ converges to x in X iff for each $i \in I$, the image filter $\pi_{i_\#}(\mathcal{L})$ converges to $\pi_i(x)$ in X_i.

Proof - ($\Rightarrow$) We know that each projection function $\pi_i : X \longrightarrow X_i$ is continuous. Also, suppose $\mathcal{L}$ is a filter on X which converges to x. "Then by preceding theorem", the image filter $\pi_{i_\#}(\mathcal{L})$ converges to $\pi_i(x)$ on X_i $\forall i \in I$.

($\Leftarrow$) For each $i \in I$, the image filter $\pi_{i_\#}(\mathcal{L})$ converges to $\pi_i(x)$ in X_i.

Let N be arbitrary but fixed nbd of x in X. By definition, $\exists$ a basis open set, say $V = \prod_{i \in I} V_i$ and $V_i = X_i$ for all i except $i = i_1, i_2, \ldots, i_n$

our word / converse part

Now $\pi_{i_{k\#}}(\mathcal{L})$ converges to $\pi_{i_k}(x)$ $\forall k = 1, 2, \ldots n$

So $V_{i_k} \in \pi_{i_{k\#}}(\mathcal{L})$

$\Rightarrow \exists F_k \in \mathcal{L}$ s.t. $V_{i_k} \supset \pi_{i_k}(F_k)$ for $k = 1, 2, \ldots n$

Note that $\pi_{i_k}^{-1}(V_{i_k}) \supset \pi_{i_k}^{-1}(\pi_{i_k}(F_k)) \supset F_k$ for $k = 1, 2, \ldots n$

$\Rightarrow N \supset V = \prod_{i \in I} V_i = \bigcap_{i=1}^{n} \pi_{i_k}^{-1}(V_{i_k}) \supset \bigcap_{k=1}^{n} F_k$.

~~But $\bigcap_{k=1}^{n} F_k$ is in $\mathcal{L}$ closed~~

But $\bigcap_{k=1}^{n} F_k \in \mathcal{L}$. since $\mathcal{L}$ is closed under finite intersection.

Therefore $N \supset \bigcap_{k=1}^{n} F_k$ is also in $\mathcal{L}$.

$\Rightarrow$ $\eta_x \subset \mathcal{L}$.

$\Rightarrow$ $\mathcal{L}$ converges to x. proved.

Ultrafilter

Def:- A filter $\mathcal{L}$ on a set X is said to be an ultrafilter if it is a maximal element in the collection of all filter on X partially ordered by inclusion, that is, $\mathcal{L}$ is an ultrafilter, if it is not properly contained in any filter on X.

$\mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots \subset \mathcal{L}$ (maximal), then $\mathcal{L}$ is an ultrafilter).

Ex:- All atomic filters whose atoms are singleton sets are maximal.

A filter $\mathcal{L}$ on X is called an ultra-filter on X iff $\mathcal{L}$ is not subset of any other filter on X.

$X=\{a,b,c\}$, $\mathcal{L}_1 = \{\{a,b\}, X\}$, $\mathcal{L}_2 = \{\{b,c\}, X\}$. since $\mathcal{L}_1$ & $\mathcal{L}_2$ are filters on X.

"Every chain has an upper bound" (Apply Zorn's lemma)

Imp

Theorem - Every filter is contained in a unique ultrafilter.

Proof - Let $\mathcal{L}$ be a filter on a set X. Let $G = \{G_\alpha\}_{\alpha \in \Delta}$ be the collection of all filter on X containing $\mathcal{L}$.

Define a binary relation '$\prec$' on G by $G_\alpha \prec G_\beta$ iff $G_\alpha \subset G_\beta$. Then the relation "$\prec$" reflexive, transitive and antisymmetric. Therefore $(G, \prec)$ is a poset.

Let $\mathcal{C} = \{\mathcal{G}_i\}_{i \in I}$ be a chain in $(G, \prec)$ s.t. $\mathcal{G}_i, \mathcal{G}_j \in \mathcal{G}$ then either $\mathcal{G}_i \subset \mathcal{G}_j$ or $\mathcal{G}_j \subset \mathcal{G}_i$.

Take $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i$

claim - (i) $\mathcal{G}$ is a filter.

(ii) $\mathcal{G}$ is an upper bound of $\mathcal{C}$.

(i) (a) clearly $\phi \notin \mathcal{G}$, because $\phi \notin \mathcal{G}_i$ for all $i \in I$.

(b) To show that $\mathcal{G}$ is closed under finite intersection.

let $A, B \in \mathcal{G} \Rightarrow A \in \mathcal{G}_i$, $B \in \mathcal{G}_j$ $\forall\, i, j \in I$

If $\mathcal{G}_i \subset \mathcal{G}_j$, then $A, B \in \mathcal{G}_j$

$\Rightarrow A \cap B \in \mathcal{G}_j$

If $\mathcal{G}_j \subset \mathcal{G}_i$, then $A, B \in \mathcal{G}_i$

$\Rightarrow A \cap B \in \mathcal{G}_i$

$\Rightarrow A \cap B \in \mathcal{G}$.

(c) Suppose $C \in \mathcal{G}$, $D \supset C \Rightarrow D \in \mathcal{G}$

Now $C \in \mathcal{G}_i$ for some $i \in I$. So $D \in \mathcal{G}_i$ as $\mathcal{G}_i$ is a filter. $\Rightarrow$ then $D \in \mathcal{G}$.

Thus $\mathcal{G}$ is a filter on X.

(ii) Every chain in $\mathcal{G}$ has an upper bound. By Zorn's lemma, it has a maximal element, say $\mathcal{H}$.

Claim :- $\mathcal{H}$ is an ultrafilter.

For this, $\mathcal{H}$ is also maximal in the set of all filters on X. Suppose $\mathcal{K}$ is a filter on X s.t. $\mathcal{H} \subset \mathcal{K}$. Then $\mathcal{L} \subset \mathcal{K}$ and so $\mathcal{K} \in \mathcal{G}$, but $\mathcal{H}$ is maximal in $\mathcal{G}$

So $\mathcal{H} = \mathcal{K}$.

Thus $\mathcal{H}$ is an ultrafilter containing $\mathcal{L}$. proved

Most imp

Pro. For a filter $\mathcal{L}$ on a set X the following statements are equivalent.

(1) $\mathcal{L}$ is an ultrafilter.

(2) For any $A \subset X$, either $A \in \mathcal{L}$ or $X - A \in \mathcal{L}$.

(3) For any $A, B \subset X$, $A \cup B \in \mathcal{L}$ iff either $A \in \mathcal{L}$ or $B \in \mathcal{L}$.

Proof (1) $\Rightarrow$ (2) Suppose $\mathcal{L}$ is an ultrafilter on X and suppose $A \subset X$.

If $A \in \mathcal{L}$, then we have nothing to proof.

If $A \notin \mathcal{L}$, then A contains no member of $\mathcal{L}$ $\Rightarrow$ each member of $\mathcal{L}$ ~~con~~ intersects $X - A$

$\Rightarrow \mathcal{L} \cup (X - A)$ has f.i.p.

$\Rightarrow \exists$ a filter $\mathcal{G}$ for which $S = \mathcal{L} \cup \{X - A\}$ is a subbase.

$\Rightarrow X - A \in \mathcal{G}$

$\Rightarrow \mathcal{G} \supset \mathcal{L}$, but $\mathcal{L}$ is an ultrafilter.

$\Rightarrow \mathcal{G} = \mathcal{L}$, $X - A \in \mathcal{G}$

$\Rightarrow X - A \in \mathcal{L}$.

(2) $\Rightarrow$ (1) Suppose, if possible $\mathcal{L}$ is not an ultrafilter on X, then {by preceding theo} $\exists$ a unique ultrafilter, say $\mathcal{G}$ containing $\mathcal{L}$.

Let $A \in \mathcal{G} - \mathcal{L}$, then $A \notin \mathcal{L}$ and so $X - A \in \mathcal{L}$

$\Rightarrow X - A \in \mathcal{G}$

Now $A, X - A \in \mathcal{G}$ and $\mathcal{G}$ is a filter.

$\Rightarrow A \cap (X - A) = \phi$, a contradicts to f.i.p. of a filter.

Thus $\mathcal{L}$ is an ultrafilter.

(3) $\Rightarrow$ (2) In view of fact $X \in \mathcal{L}$, so if we choose $B = X - A$, then

$A \in \mathcal{L}$ or $X - A \in \mathcal{L}$.

(2) $\Rightarrow$ (3) ($\Rightarrow$) Suppose either $A \in \mathcal{L}$ or $B \in \mathcal{L}$. then $A \cup B \supset A$ as well as B

$\Rightarrow A \cup B \in \mathcal{L}$ {by superset property}

($\Leftarrow$) Assume that $A \cup B \in \mathcal{L}$. then we have to show that either $A \in \mathcal{L}$ or $B \in \mathcal{L}$.

Suppose on the contrary, neither $A \in \mathcal{L}$ nor $B \in \mathcal{L}$.

$\Rightarrow$ ~~$X - A \in \mathcal{L}$~~ $X - A, X - B \in \mathcal{L}$

$\Rightarrow (X - A) \cap (X - B) \in \mathcal{L}$

$\Rightarrow X - (A \cup B) \in \mathcal{L}$. a contradiction.

So that either $A \in \mathcal{L}$ or $B \in \mathcal{L}$.

Hence proved

with ultrafilter case converges $\Leftrightarrow$ cluster.

Imp

Pro. An ultrafilter converges to a point iff that point is a cluster point of $\mathcal{L}$.

Proof – ($\Rightarrow$) Suppose $\mathcal{L}$ converges to x, then

$\eta_x \subset \mathcal{L} \Rightarrow N \cap F \neq \phi \;\; \forall N \in \eta_x$ & $F \in \mathcal{L}$

$\Rightarrow \mathcal{L}$ clusters at x.

($\Leftarrow$) Suppose $\mathcal{L}$ clusters at x. If $\mathcal{L}$ does not converges to x, then $\exists\, N \in \eta_x$ s.t. $N \notin \mathcal{L}$

$\Rightarrow X - N \in \mathcal{L}$, but x is a cluster point of $\mathcal{L}$.

$\Rightarrow$ every nbd of x intersects every member of $\mathcal{L}$.

Where $N \cap (X - N) = \phi$. a contradiction.

Thus $\mathcal{L}$ converges to x.

proved.

Theorem - X is compact iff every ultrafilter converges in X.

Proof - ($\Rightarrow$) Suppose X is compact and $\mathcal{L}$ is an ultrafilter in X. So $\mathcal{L}$ is a filter in X. By compactness of X, $\mathcal{L}$ has a cluster point in X, say α. But an ultrafilter converges to a point in X iff that point is a cluster point of $\mathcal{L}$.

Thus $\mathcal{L}$ converges to α in X.

($\Leftarrow$) Suppose every ultrafilter $\mathcal{L}$ ~~in X~~ converges in X. In particular, filter $\mathcal{L}$ converges in X. This implies that $\mathcal{L}$ clusters in X.

$\Rightarrow$ X is compact. {$\because$ X is compact $\Leftrightarrow$ every filter on X has a cluster point in X}

proved.

Tychonoff Theorem :- Let $\{X_i : i \in I\}$ be a collection of non-empty spaces and let $X \left(= \prod_{i \in I} X_i\right)$ be its topological product. Then X is compact iff each X_i is compact.

Proof ($\Rightarrow$) Each projection function $\pi_i : X \longrightarrow X_i$ is continuous and it is well known that continuous image of a compact space is compact.

it follows that $X_i = \pi_i(X)$ is compact for each $i \in I$.

($\Leftarrow$) Suppose each component space X_i is compact. We have to show that X is compact, we show that every ultrafilter is convergent in X.

Suppose $\mathcal{L}$ is an ultrafilter on X. then

$\mathcal{L}_i = \pi_{i\#}(\mathcal{L})$ is a filter in X_i

claim - ~~$\mathcal{L}_i = \pi_{i\#}$~~ $\mathcal{L}_i$ is indeed an ultrafilter on

Let $A \subset X_i$, put $B = \pi_i^{-1}(A) \subset X$

$\Rightarrow X - B = \pi_i^{-1}(X - A) \subset X$

$\Rightarrow$ either $B \in \mathcal{L}$ or $X - B \in \mathcal{L}$, because $\mathcal{L}$ is an ultrafilter on X.

Case I. If $B \in \mathcal{L}$, then $A = \pi_i(B) \in \pi_{i\#}(\mathcal{L}) \subset \pi_{i\#}(\mathcal{L}) = \mathcal{L}_i$

$\Rightarrow A \in \mathcal{L}_i$

Case II. If $X - B \in \mathcal{L}$, then $\pi_i(X - B) = X_i - A \in \pi_{i\#}(\mathcal{L}) =$

$\Rightarrow X_i - A \in \mathcal{L}_i$

$\Rightarrow \mathcal{L}_i$ is an ultrafilter on X_i and that X_i is compact

$\Rightarrow \mathcal{L}_i$ converges to a point in X_i, say at $x_i = \pi_i(x)$

$\forall\, i \in I$ and for some $x \in X$.

$\Rightarrow$ But we know that, for each $i \in I$, the filter $\pi_i^{\#}(\mathcal{F})$ converges to $\pi_i(x)$ in X_i, then $\mathcal{F}$ converges to x

i.e. $\mathcal{F}$ converges to x

$\Rightarrow \mathcal{F}$ clusters at $x \in X$.

$\Rightarrow X$ is compact. proved.

Universal - A net S in a set X is said to be universal if for each subset A of X, S is either eventually in A or eventually in $X - A$.

Homotopy

"path is a continuous function"

Def - If f and f' are continuous maps of the space X into the space Y, we say that f is homotopic to f' if there exist is a continuous map $F: X \times I \longrightarrow Y$ such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = f'(x)$$

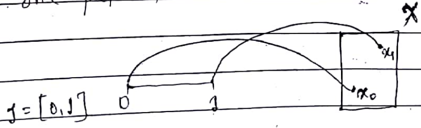
for each x. (Here $I = [0,1]$). The map F is called a homotopy between f and f'. If f is homotopic to f', we write $f \overset{F}{\simeq} f'$.

If $f \overset{F}{\simeq} f'$ and f' is a constant map, then f is nulhomotopic.

Homotopy between Paths

~~Two path~~

Path - If $f: [0,1] \to X$ is a continuous map s.t. $f(0) = x_0$ and $f(1) = x_1$, then f is a path in X from x_0 to x_1, where x_0 is the initial point and x_1 is the final point (terminal point) of the path f.



$\{f: I \longrightarrow X\}$

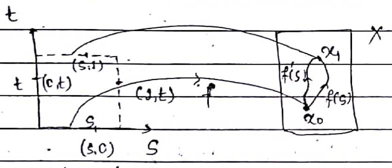
$f(0) = x_0$, $f(x_1) = x_1$

Homotopic path – Two path f and f' mapping the interval $I = [0,1]$ into X, are said to be path homotopic if they have the same initial point x_0 and the same final point x_1 and if there is a continuous map $F : I \times I \to X$ s.t.

$$F(s,0) = f(s) \text{ and } F(s,1) = f'(s), \quad 0 \le s \le 1$$

$$F(0,t) = x_0 \text{ and } F(1,t) = x_1, \quad 0 \le t \le 1$$

for each $s \in I$ and each $t \in I$. We call F a path homotopy between f and f', denoted by $f \simeq_p f'$



st - plane

Imp

Lemma – The relation $\simeq$ and $\simeq_p$ are equivalence relation.

Proof – Let us verify the properties of an equivalence relation.

Let $\mathcal{L} = \{ f \mid f : X \xrightarrow{\text{continuous}} Y \}$

① Reflexivity – Define $F : X \times I \to Y$ by $F(x,t) = f(x)$ $\forall t \in I$

(i) F is continuous

(ii) $F(x,0) = f(x)$ and $F(x,1) = f'(x)$

(i) Since each f is continuous, so F is continuous.

(ii) $F(x,0) = f(x)$, $F(x,1) = f'(x)$ {by definition}

i.e. $\forall f \in \mathcal{L} \Rightarrow f \overset{F}{\simeq} f$

② Symmetry –

$f \simeq g \Rightarrow \exists$ a continuous map $F : X \times I \to Y$, s.t.

$$F(x,0) = f(x), \quad F(x,1) = g(x)$$

Define $G : X \times I \to Y$ by $G(x,t) = F(x, 1-t)$

clearly, G is continuous and

$$\left.\begin{aligned} G(x,0) &= F(x,1) = g(x) \\ G(x,1) &= F(x,0) = f(x) \end{aligned}\right\} \Rightarrow g \simeq f$$

i.e. $f \overset{F}{\simeq} g \Rightarrow g \overset{F}{\simeq} f$

Transitivity -

Let $f \simeq g$ & $g \simeq h$. We show that $f \simeq h$.

Let F be a homotopy path between f and g, and F' be a homotopy path between g and h.

Define $G: X \times I \longrightarrow Y$ by

$$G(x,t) = \begin{cases} F(x, 2t), & \text{for } t \in [0, 1/2] \\ F'(x, 2t-1), & \text{for } t \in [1/2, 1] \end{cases}$$

If $t = \frac{1}{2}$, then $F(x, 2t) = F(x,1) = f'(x)$

and $F'(x, 2t-1) = F'(x, 0) = f'(x)$.

Because G is continuous on the closed subset $X \times [0, 1/2]$ & $X \times [1/2, 1]$ on $X \times I$. It is continuous on all $X \times I$.

So G is the required homotopy between f and h.

proved.

Transitivity :-

$f \simeq g$, $g \simeq h \Rightarrow \exists$ homotopies F and G s.t.

$F: X \times I \longrightarrow Y$ and $G: X \times I \longrightarrow Y$

F and G both are continuous.

$F(x,0) = f(x)$, $F(x,1) = g(x)$

$G(x,0) = h(x)$, $G(x,1) = h(x)$, $\forall x \in X$

Define $H: X \times I \longrightarrow Y$ by

$$H(x,t) = \begin{cases} F(x, 2t), & 0 \le t \le \frac{1}{2} \\ G(x, 2t-1), & \frac{1}{2} \le x \le 1. \end{cases}$$

Observation -

(i) By (i) F and G are continuous, so that H is also continuous.

(ii) $H(x,0) = F(x,0) = f(x)$

$H(x,1) = G(x,1) = h(x) \quad \forall x \in X$

(i) Pasting lemma - (a) $X = A \cup B$, $f: A \xrightarrow{\text{continuous}} Y$

$g: B \xrightarrow{\text{continuous}} Y$

(b) A and B are closed subset of X

(c) $f(x) = g(x) \quad \forall x \in A \cap B$

Define $h: X \longrightarrow Y$ by

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

Set $A = X \times [0, 1/2]$, $B = X \times [1/2, 1]$

then

$A \cup B = (X \times [0, 1/2] \cup X \times [1/2, 1])$

$= X \times ([0, \frac{1}{2}] \cup [1/2, 1])$

$= X \times I = [0, 1]$.

$\Rightarrow h$ is continuous

always defined. ~~path homotopy~~ classes is not

(b) clearly $A = [0, 1/2]$ & $B = [1/2, 1]$ are closed subset of X

(c) At $(x, 1/2)$

$$F(x, 1) = g(x) = G(x, 0)$$

Thus $f \simeq g,\ g \simeq h \Rightarrow f \simeq h.$

proved.

{ Every class either $[f] \cap [g] = \phi$ (disjoint) or $[f] = [g]$ ~~are~~ (identical) }

Imp

Exa - Let f and g are any two maps of a space X into R^2 (i.e. $f, g : I \to Y$), then f and g are homotopic.

The map $F : X \times I \to Y$ defined by

$$F(x, t) = (1-t) f(x) + t g(x) \quad \{\text{scalar}\}$$

$$= ((1-t) f + t g)(x).$$

(1) F is continuous

(2) $F(x, 0) = f(x), \quad F(x, 1) = g(x)$

F is homotopy between f and g ($f \overset{F}{\simeq} g$).

It is called a 'straight line homotopy" because it moves the point $f(x)$ and $g(x)$

along the straight-line segment joining them.

If f and g are paths from x_0 to x_1, then F will be a path homotopy.

Exa - Let $f : I \to \mathbb{R}^2$, $g : I \to \mathbb{R}^2$, $h : I \to \mathbb{R}^2$ ~~the~~ defined by

$f(t) = (\cos \pi t, \sin \pi t)$

$g(t) = (\cos \pi t, 2 \sin \pi t)$

$h(t) = (\cos \pi t, -\sin \pi t)$

Then $f \simeq_p g,\ g \simeq_p h$ #

Exa - Let X denote the punctured plane, $R^2 - \{0\}$ which we shall denote by $R^2 - 0$ for short. The following paths in X

$f(t) = (\cos \pi t, \sin \pi t)$

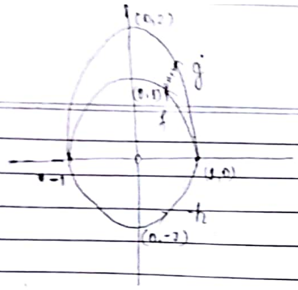
$g(t) = (\cos \pi t, \sin \pi t)$

$\{ f \not\simeq_p h, \ g \not\simeq_p h \}$

are path homotopic. The straight-line homotopy between them is an acceptable path homotopy. But the straight-line homotopy between f and the path

$$h(t) = (\cos \pi t, -\sin \pi t)$$

is not acceptable. for its image does not lie in the space $X = R^2 - 0$.



Product- If f is a path in X from x_0 to x_1 and if g is a path in X from x_1 to x_2 we define the product $f*g$ of f and g to be the path h given by

$$h(t) = \begin{cases} f(2t), & \text{for } t \in [0, \frac{1}{2}] \\ g(2t-1), & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

The function h is well-defined and continuous, by pasting lemma. h is a path from x_0 to x_2.

Path homotopy classes, $[f]*[g] = [f*g]$

The operation $*$ has the following properties:

1. Associativity -

$$([f]*[g])*[h] = [f]*([g]*[h])$$

2. Right and left identities: Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then

$$[f]*[e_{x_1}] = [f], \quad \& \quad [e_{x_0}]*[f] = [f]$$
(right) (left)

3. Inverse - Let f is a path in X from x_0 to x_1 and let $\tilde{f}$ be the path define by $\tilde{f}(t) = f(1-t)$. It is called the reverse of f. Then

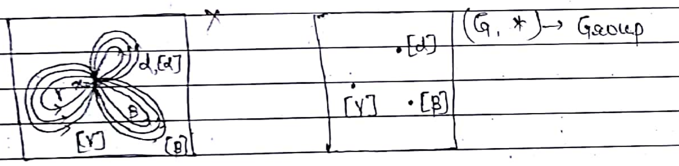
$$[f]*[\tilde{f}] = [e_{x_0}] \quad \& \quad [\tilde{f}]*[f] = [e_{x_1}]$$
(right) (left)

* 1. If $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic

Imp 2. A space X is said to be contractible if the identity map $i_X : X \to X$ is nulhomotopic.

3. The product of two path-homotopy classes is not always defined.

First Fundamental group :- Let X be a space and let $x_0 \in X$. A path in X that begins and ends at x_0 is called a loop based at x_0. The set of path homotopy classes of loops based at x_0 with the operation $*$ is called the fundamental group of X relative to the base point x_0. It is denoted by $\pi_1(X, x_0)$.



Definition - Let $h : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map. Define by $h(x_0) = y_0$

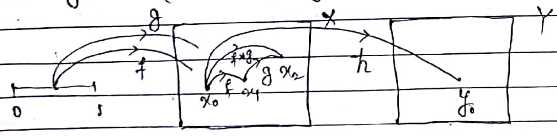
$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

by the equation $h_*([f]) = [h \circ f]$

The map h_* is called the homomorphism induced by h, relative to the base point x_0.

Theorem - h- hat is a homomorphism.

Proof - $h \circ (f * g) = (h \circ f) * (h \circ g)$



Let $f * g : I \longrightarrow X$, $h : X \longrightarrow Y$, then

$h \circ (f * g) : I \longrightarrow Y$ be a map define by

$f(0) = x_0$, $f(1) = x_1$

$g(0) = x_1$, $g(1) = x_2$

$(f * g)(0) = x_0$, $(f * g)(1) = x_2$

Then $h \circ (f * g)(0) = h((f * g)(0)) = h(x_0) = y_0 \in Y$

and $((h \circ f) * (h \circ g))(0) = (h \circ f)(0) * (h \circ g)(0)$

$= h(f(0)) * h(g(0))$

$= h(x_0) * h(x_1)$

So that $h \circ (f * g) = (h \circ f) * (h \circ g)$

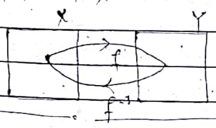
$h_*([f * g]) = [h \circ (f * g)] = [(h \circ f) * (h \circ g)]$

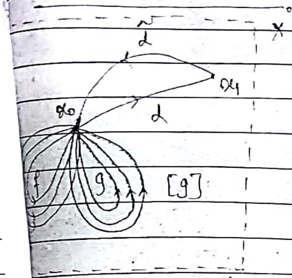
$= [h \circ f] * [h \circ g]$

$= h_*([f]) * h_*([g])$

Thus h_* is a homomorphism.

$f : X \to Y$ is bijective iff $f \cdot f^{-1} = I_X$, $f^{-1} f = I_Y$





Definition – Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\tilde{\alpha}] * [f] * [\alpha]$$

$$= [(\tilde{\alpha} * f)] * [\alpha]$$

$$= [(\tilde{\alpha} * f) * \alpha]$$

~~Whether $\hat{\alpha}$ is an isomorphism?~~

The map $\hat{\alpha}$, which we call "α-hat" is well-defined because the operation $*$ is well defined.

Theorem Imp – The map $\hat{\alpha}$ is a group ~~homo~~isomorphism.

Proof – To show that $\hat{\alpha}$ is a homomorphism, we compute

$$\hat{\alpha}([f] * [g]) = \hat{\alpha}([f]) * \hat{\alpha}([g])$$

Now $\hat{\alpha}([f]) * \hat{\alpha}([g]) = ([\tilde{\alpha}] * [f] * [\alpha]) * ([\tilde{\alpha}] * [g] * [\alpha])$

$$= [\tilde{\alpha}] * [f] * ([\alpha] * [\tilde{\alpha}]) * [g] * [\alpha]$$

$$= [\tilde{\alpha}] * [f] * [e_{x_0}] * [g] * [\alpha]$$

$$= [\tilde{\alpha}] * [f] * ([e_{x_0}] * [g]) * [\alpha]$$

$$= [\tilde{\alpha}] * ([f] * [g]) * [\alpha]$$

$$= [\tilde{\alpha}] * [f * g] * [\alpha]$$

$$= \hat{\alpha}([f * g])$$

To show that $\hat{\alpha}$ is an isomorphism. Denoting $\beta = \tilde{\alpha}$, then we have to show that

$$\hat{\alpha}\hat{\beta} = I_{\pi_1(X, x_1)}, \quad \hat{\beta}\hat{\alpha} = I_{\pi_1(X, x_0)}$$

For any $[h] \in \pi_1(X, x_1)$

$$\hat{\alpha}\hat{\beta}[h] = \hat{\alpha}[\hat{\beta}([h])] = \hat{\alpha}([\tilde{\beta}] * [h] * [\beta])$$

$= [\tilde{\alpha}] * ([\tilde{\beta}] * [h] * [\beta]) * [\alpha]$

$= ([\beta] * [\tilde{\beta}]) * [h] * ([\tilde{\alpha}] * [\alpha])$

$= [\beta * \tilde{\beta}] * [h] * [\tilde{\alpha} * \alpha]$

$= [e_{x_1}] * ([h] * [e_{x_1}])$

$= [e_{x_1}] * [h] = [e_{x_1} * h]$

$= [h] = I_{\pi_1(X, x_1)}([h])$

$= I_{\pi_1(X, x_1)}$

For any $[f] \in \pi_1(X, x_0)$

$\hat{\beta}\hat{\alpha}[f] = \hat{\beta}[\hat{\alpha}([f])] = \hat{\beta}([\tilde{\alpha}] * [f] * [\alpha])$

$= [\tilde{\beta}] * ([\tilde{\alpha}] * [f] * [\alpha]) * [\beta]$

$= ([\tilde{\beta}] * [\beta]) * [f] * ([\alpha] * [\tilde{\alpha}])$

$= [\tilde{\beta} * \beta] * [f] * [\alpha * \tilde{\alpha}]$

$= [e_{x_1}] * ([f] * [e_{x_1}])$

$= [e_{x_1}] * [f] = [e_{x_1} * f]$

$= [f] = I_{\pi_1(X, x_0)}([f])$

$= I_{\pi_1(X, x_0)}$ proved

Imp

Simply Connected Space :- A topological space X is said to be simply connected if

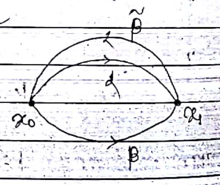
① X is a connected space

② for an $x_0 \in X$, the first fundamental group $\pi_1(X, x_0)$ is a trivial group, i.e. $\pi_1(X, x_0)$ contains only one member and hence at each $x_0 \in X$, $\pi_1(X, x_0)$ is a trivial group.

Imp

Theorem - If X is a simply connected space, then any two paths joining the same initial and final points are homotopic.

Proof - Let α and β be two path from x_0 to x_1. $\alpha \circ \tilde{\beta}$ is a loop based at $x_0 \in X$. By simply connectedness of X, then loop $\alpha \circ \tilde{\beta}$ is homotopic to constant path e_{x_0}.



In the context, we have

$([\alpha] * [\tilde{\beta}]) * [\beta] = [e_{x_0}] * [\beta]$
$= [e_{x_0} * \beta] = [\beta]$

and $[\alpha] * ([\tilde{\beta}] * [\beta]) = [\alpha] * [e_{x_1}]$
$= [\alpha * e_{x_1}] = [\alpha]$

So that $[\alpha] = [\beta]$

$\Rightarrow \alpha \in [\beta]$ $\{\because b \in [a] \Leftrightarrow [a] = [b]\}$

$\Rightarrow \boxed{\alpha \simeq_p \beta}$ Proved.

Theorem – If $h : (X, x_0) \longrightarrow (Y, y_0)$ and $k : (Y, y_0) \longrightarrow (Z, z_0)$ are continuous, then

(i) $(k \circ h)_* = k_* \circ h_*$

(ii) If $i : (X, x_0) \longrightarrow (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Proof – We have $k \circ h : (X, x_0) \longrightarrow (Z, z_0)$. Also

$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

and $k_* : \pi_1(Y, y_0) \longrightarrow \pi_1(Z, z_0)$

and $(k \circ h)_* : \pi_1(X, x_0) \longrightarrow \pi_1(Z, z_0)$

Then for any $[f] \in \pi_1(X, x_0)$,

$(k \circ h)_*([f]) = [(k \circ h) \circ f]$
$= [k \circ (h \circ f)]$
$= k_*([h \circ f])$
$= k_*(h_*([f]))$
$= (k_* \circ h_*)([f])$

Because $[f]$ was an arbitrary element of $\pi_1(X, x_0)$, it follows that

$\boxed{(k \circ h)_* = k_* \circ h_*}$: proved – I.

Similarly, $i_*([f]) = [i \circ f] = [f]$

proved.

Theorem – ~~Supp~~ If $h : (X, x_0) \longrightarrow (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

Proof :- We know that h_* is a homomorphism (proof – 118)

In order to show that h_* is an isomorphism, it suffices to show that h_* is invertible.

Since h is homeomorphism $\Rightarrow h^{-1}$ is exists.

and $h^{-1} : (Y, y_0) \longrightarrow (X, x_0)$

Take $k = h^{-1}$, then

$k_* \circ h_* = (k \circ h)_* = (h^{-1} \circ h) = i_*$, where i is the identity mapping on X.

and $h_* \circ k_* = (h \circ k)_* = (h \circ h^{-1}) = j_*$, where j is the identity mapping on Y.

Thus $k_* \circ h_* = i_*$, $h_* \circ k_* = j_*$

$\Rightarrow k_*$ is the inverse of h_*

$\Rightarrow h_*$ is invertible.

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$\Rightarrow h_*$ is one-one onto

So that h_* is an isomorphism. proved.

*** Covering Space :-** Let $p: E \rightarrow B$ be a continuous surjective map. Let $U \subset B$, then

Def ① U said to be evenly covered by p if the inverse image $p^{-1}(U)$ give rise a class $\{V_\alpha\}_{\alpha \in J}$ such that each V_α is open in E and all members of this family are mutually disjoint. i.e. $V_\alpha, V_\beta \in \{V_\alpha\}_{\alpha \in J} \Rightarrow V_\alpha \cap V_\beta = \phi$ and $p^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha$.

The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into slices.

Def ② Let $p: E \rightarrow B$ be continuous and surjective. Then p is said to be covering map, if for each $b_0 \in B$ and any open nbd U of b_0 is evenly covered by p, and E is said to be a covering space.

* ① "If $p: E \rightarrow B$ is a covering map" then for each $b \in B$ the subspace $p^{-1}(b)$ of E has the discrete topology. (" then p is an open map")

① For each slice V_α is open in E and intersects the set $p^{-1}(b)$ in a single point, therefore this point is open in $p^{-1}(b)$.

Exa - ① Let X be any space; let $i: X \rightarrow X$ be the identity map. Then i is a covering map. (of the most trivial sort)

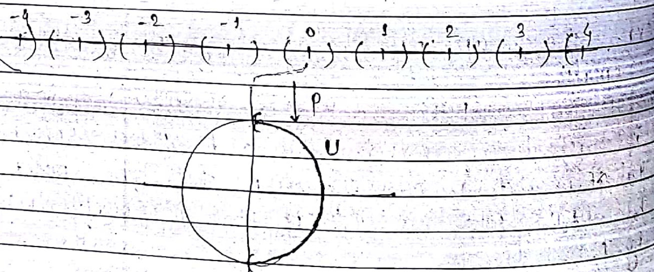
② The map $p: E \rightarrow X$ given by $p(x, i) = x$ is again a (rather trivial) covering map.

Imp

Theorem - The map $p: \mathbb{R} \rightarrow S^1$ given by the equation
$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$
is a covering map.

Proof - Let $U \subset S^1$ consisting of those points having positive first coordinate. The set $p^{-1}(U)$ consists of those points x for which $\cos 2\pi x$ is positive. i.e. it is union of the intervals

$$V_n = \left(n - \frac{1}{4}, n + \frac{1}{4}\right) \quad \text{for all } n \in \mathbb{Z}$$



Now, restricted to any closed set interval $\bar{V}_n$ the map p is injective because $\sin 2\pi x$ is strictly monotonic on each an interval.

Furthermore, p carries $\overline{V_n}$ surjectively onto $\overline{U}$ and V_n to U, by intermediate value theorem. Since $\overline{V_n}$ is compact, $p/\overline{V_n}$ is a homeomorphism of $\overline{V_n}$ with $\overline{U}$ $\Rightarrow$ p/V_n is a homeomorphism of V_n with U.

Similar arguments can be applied to the intersection of S^1 with the upper and lower open half-planes and with open left-hand half-plane. These open sets cover S^1 and each of them is evenly covered by p. Hence $p: R \to S^1$ is a covering map.

Defn/- If $p: E \to B$ is a covering map, then p is a local homeomorphism of E with B. i.e. each point e of E has a nbd that is mapped homeomorphically by p onto an open subset of B.

Exg/- The map $p: R_+ \to S^1$ given by the equation

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is surjective and it is a local homeomorphism. But it is not a covering map, for the point $b_0 = (1,0)$ has no nbd U that is evenly covered by p.

Exg -/ The map $p: S^1 \to S^1$ is given by

$$p(z) = z^2$$

is a covering map.

Theorem -/ If $p: E \to B$ and $p': E' \to B'$ are covering maps, then

~~$p\times p': E \to E'$~~ $p\times p': E\times E' \longrightarrow B\times B'$

is a covering map.

Proof - Given $b \in B$ and $b' \in B'$, let U and U' be nbd's of b and b' respectively, that are evenly covered by p and p', respectively.

Let $\{V_\alpha\}_{\alpha\in I}$ and $\{V_\beta\}_{\beta\in J}$ be partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$ respectively, into slices. Then a family of slices $\{V_\alpha \times V_\beta\}$ in $E\times E'$ such that

$$(p\times p')^{-1}(U\times U') = \bigcup_{\substack{\alpha\in I\\ \beta\in J}} V_\alpha \times V_\beta$$

$$\Rightarrow \quad p^{-1}(U) \times (p')^{-1}(U') = \left(\bigcup_{\alpha\in I} V_\alpha\right) \times \left(\bigcup_{\beta\in J} V_\beta\right)$$

Example - Consider the space $T = S^1 \times S^1$, it is called the torus. The product map $p\times p: R\times R \longrightarrow S^1\times S^1$ is a covering of the torus by the plane R^2.

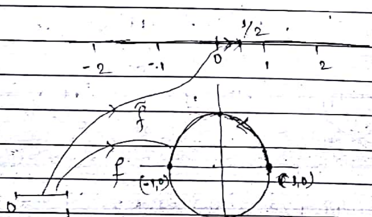
Lifting :- Let $p: E \to B$ be a map. If f is a continuous mapping of some space X into B, a lifting of f is a map $\tilde{f}: X \to E$ such that $p \circ \tilde{f} = f$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Exa - Let $p: \mathbb{R} \to S^1$ be covering map defined by

$p(s) = (\cos 2\pi s, \sin 2\pi s)$ $\quad (I = [0,1])$

The path $f: I \to S^1$ be given by $f(s) = (\cos \pi s, \sin \pi s)$ lifts to the path $\tilde{f}(s) = \frac{s}{2}$ begining at 0 and ending at $\frac{1}{2}$



Verification :

for any $s \in I$

$(p \circ \tilde{f})(s) = p(\tilde{f}(s)) = p(\frac{s}{2})$

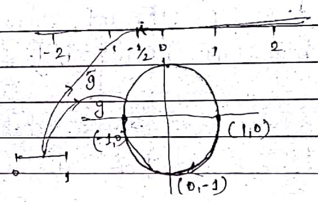
$= (\cos 2\pi \frac{s}{2}, \sin 2\pi \frac{s}{2}) = (\cos \pi s, \sin \pi s)$

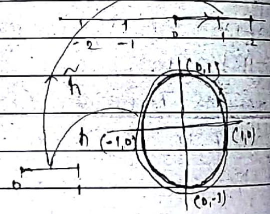
$= f(s)$

$\Rightarrow \quad p \circ \tilde{f} = f$.

And the path $g(s) = (\cos \pi s, -\sin \pi s)$ lifts to the path $\tilde{g}(s) = -\frac{s}{2}$ begining at 0 and ending at $-\frac{1}{2}$.

The path $h(s) = (\cos 4\pi s, \sin 4\pi s)$ lifts to the path $\tilde{h}(s) = 2s$ begining at 0 and ending at 2.





Theorem- (Fundamental theorem of Circle) - Let $p: E \to B$ be a covering map, let $p(e_0) = b_0$. Any path $f: [0,1] \to B$ begining at b_0 has a unique lifting to a path $\tilde{f}$ in E begining at e_0.

Proof - Cover B by open sets U each of which is evenly covered by p.

Let $P = \{0 = s_0, s_1, s_2, \dots, s_n = 1\}$ be a partition $[0,1]$ such that for each i, $f([s_i, s_{i+1}])$ lies in such an open set U. (By lebesgue number lemma) the lifting

Define $\tilde{f} : [0,1] \to E$ step by step.

First, define $\tilde{f}(0) = e_0$. Then supposing $\tilde{f}(s)$ is defined for $0 \le s \le s_i$, we define $\tilde{f}$ on $[s_i, s_{i+1}]$ as follows: The set $f([s_i, s_{i+1}])$ lies in some open set U that is evenly covered by p.

Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices each set V_α is mapped homeomorphically onto U by p.

Now $\tilde{f}(s_i)$ lies in one of these sets, say in V_0. Define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$ by

$$\tilde{f}(s) = (p/V_0)^{-1}(f(s))$$

Because $p/V_0 : V_0 \to U$ is a homeomorphism. $\tilde{f}$ will be continuous on $[s_i, s_{i+1}]$

Continuing in this way, we define $\tilde{f}$ on $[0,1]$, continuity of $\tilde{f}$ follows from the pasting lemma.

$p \circ \tilde{f} = f$ (by definition of $\tilde{f}$)

The uniqueness of $\tilde{f}$ -

Definition - Let $p : E \to B$ be a covering map. let $b_0 \in B$ choose e_0 so that $p(e_0) = b_0$. Given an element of $\pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then

$$\phi : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

We call ϕ the lifting correspondence derived from the covering map. Where ϕ is a well defined. It depends of course on the choice of the point e_0.

Theorem - Let $p : E \to B$ be a covering map, let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence

$$\phi : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

is surjective. If E is simply connected then ϕ is bijective.

Proof - (1) Let f be a loop based at b_0 and let $\tilde{f}$ be its lifting which begin at e_0 and ends at $e_1 \in p^{-1}(b_0)$. Then $f = p \circ \tilde{f}$

For any given $e_1 \in p^{-1}(b_0)$. $\exists$ a loop ~~of~~ f based at b_0 i.e. $[f] \in \pi_1(B, b_0)$ s.t. $\phi([f]) = e_1$

$\Rightarrow \phi$ is surjective.

(ii) Suppose E is simply connected. Let $[f], [g] \in \pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$

i.e. $\tilde{f}(1) = \tilde{g}(1)$

Let $\tilde{f}$ and $\tilde{g}$ be the lifting of f and g respectively to paths in E that begin at e_0 then $\tilde{f}(1) = \tilde{g}(1)$

Since E is simply connected, there is a path F in E between $\tilde{f}$ and $\tilde{g}$. ~~Then~~ i.e.

~~(a) $p \circ F$ is~~ $\tilde{f} \overset{F}{\simeq}_p \tilde{g}$

claim - $\tilde{f} \overset{p \circ F}{\simeq}_p \tilde{g}$

(a) $p \circ F$ is continuous.

(b)

(i) $(p \circ F)(s,0) = p(F(s,0)) = p(\tilde{f}(s)) = \tilde{f}(s)$

$(p \circ F)(s,1) = p(F(s,1)) = p(\tilde{g}(s)) = \tilde{g}(s)$

(ii) $(p \circ F)(0,t) = p(F(0,t)) = p(e_0) = e_0$

$(p \circ F)(1,t) = p(F(1,t)) = p(e_1) = e_1$

//.

Imp

Theorem - The fundamental group of S^1 is isomorphic to the additive group of integers.

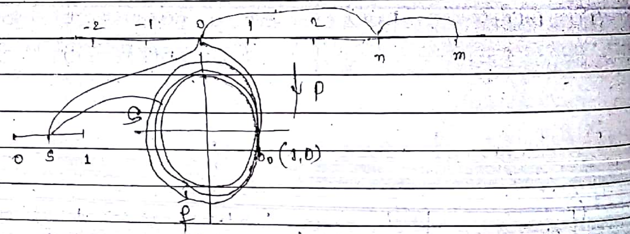
Proof - Let $p: \mathbb{R} \to S^1$ be a covering map defined by

$p(x) = (\cos 2\pi x, \sin 2\pi x)$

~~where R is simply connected.~~

Let $e_0 = 0$ and $p(e_0) = b_0$, where $b_0 = (1,0)$

Then $p^{-1}(b_0) = \{\cdots -2, -1, 0, 1, 2, \cdots\} = \mathbb{Z}$



Since R is simply connected, the lifting correspondence

$\phi: \pi_1(S^1, b_0) \longrightarrow \mathbb{Z}$

is bijective.

Claim - ϕ is homomorphism i.e. $\phi([f] * [g]) = \phi([f]) + \phi([g])$

Let $[f], [g] \in \pi_1(S^1, b_0)$. Let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths on R beginging at n.

Let $\tilde{f}(1) = n$ and $\tilde{g}(1) = m$

then $\phi([f]) = n$ and $\phi([g]) = m$

Define $\tilde{\tilde{g}}(x) = n + \tilde{g}(x)$

$\tilde{\tilde{g}}(0) = n + \tilde{g}(0) = n + 0 = n$

$\tilde{\tilde{g}}(1) = n + \tilde{g}(1) = n + m$

(1) $p(n+x) = p(x)$ $\left\{ \begin{array}{l} \because \cos 2\pi(n+x) = \cos 2\pi x \\ \sin 2\pi(n+x) = \sin 2\pi x \end{array} \right\}$

(2) $g \longrightarrow \tilde{\tilde{g}}$

$(p \circ \tilde{\tilde{g}})(s) = p(\tilde{\tilde{g}}(s)) = p(n + \tilde{g}(s)) = p(\tilde{g}(s))$

$= (p \circ \tilde{g})(s) \quad \forall \ s \in I$

$\Rightarrow p \circ \tilde{\tilde{g}} = p \circ \tilde{g} = g$

i.e. $g \longrightarrow \tilde{\tilde{g}}$

(3) $f * g \longrightarrow \tilde{f} * \tilde{\tilde{g}}$

$(p \circ (\tilde{f} * \tilde{\tilde{g}}))(s) = ((p \circ \tilde{f}) * (p \circ \tilde{\tilde{g}}))(s)$

$= (p \circ \tilde{f})(s) * (p \circ \tilde{\tilde{g}})(s)$

$= p(\tilde{f}(s)) * p(\tilde{\tilde{g}}(s))$

$= (p \circ \tilde{f})(s) * p(n + \tilde{g}(s))$

$= (p \circ \tilde{f})(s) * p(\tilde{g}(s))$

$= (p \circ \tilde{f})(s) * (p \circ \tilde{g})(s)$

$= ((p \circ \tilde{f}) * (p \circ \tilde{g}))(s)$

$= (f * g)(s)$

$\Rightarrow (p \circ (\tilde{f} * \tilde{\tilde{g}})) = (f * g)$

i.e. $f * g \longrightarrow \tilde{f} * \tilde{\tilde{g}}$

Now $\phi([f] * [g]) = \phi([f * g])$

$= (\tilde{f} * \tilde{\tilde{g}})(1) = n + m$

$= \tilde{f}(1) + \tilde{g}(1)$

$= \phi([f]) + \phi([g])$

Hence ϕ is homomorphism.

Retraction/- If $A \subset X$, a retraction of X onto A is a continuous map $r: X \longrightarrow A$ such that $r|_A$ is the identity of map. A. If such a map r exists, we say that A is a retract of X.

Lemma 1/ Let $r: X \longrightarrow A$ be a rectraction map of A, then the homomorphism of fundamental groups induced by ~~incute~~ inclusion $j: A \longrightarrow X$ is injective.

Proof- Since $r: X \longrightarrow A$ is a retraction, and let $j: A \longrightarrow X$ be a injective by map defined by $j(a) = a \quad \forall\ a \in A$

then $roj: A \longrightarrow A$ be identity map.

$$(roj)(a) = r(j(a)) = r(a) = a = I_A(a)$$

$$\Rightarrow roj = I_A$$

$$\Rightarrow (roj)_* = r_* \circ j_*$$

$$\Rightarrow j_* \text{ is injective}$$

proved/

Lemma 2. $\nexists$ any retraction map from B^2 to S^1.

$B^2 = \{(x_1, x_2) \in R^2 \mid x_1^2 + x_2^2 \leq 1\}$

or There is no retraction of B^2 onto S^1.

Lemma- Let $h: S^1 \longrightarrow X$ be a continuous map. Then the following condition are equivalent

1. h is nullhomotopic
2. h extends to a continuous map $k: B^2 \longrightarrow X$
3. h_* is the trivial homomorphic of fundamental groups (on S^1)

The inclusion map $j: S^1 \longrightarrow R^2 - 0$ is not nullhomotopic.

identity mapping $i: S^1 \longrightarrow S^1$ is not nullhomotopic

~~imp~~ ~~imp~~

Fundamental Theorem of algebra - A polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 = 0$$

of degree $n > 0$ with real or complex coefficients has at least one (real or complex) root.

Proof - Step I. Consider the map $f: S^1 \longrightarrow S^1$ defined by $f(z) = z^n$, $z \in C$

Let $P_0: I \longrightarrow S^1$ be the standard loop in S^1.

$p_0(s) = e^{2\pi i s} = (\cos 2\pi s, \sin 2\pi s)$

Its image under f_* is the loop.

$(f \circ p_0)(s) = e^{(2\pi i s)n} = (\cos 2\pi n s, \sin 2\pi n s)$

this loop lifts to the path $s \to ns$ in the space covering space R.

Thus f_* is "multiplication by n" in the fundamental group of S^1 so that f_* is injective.

Step II. We show that if $g: S^1 \to R^2 - 0$ is the map $g(z) = z^n$, then g is not nullhomotopic.

Assume $j: S^1 \to R^2 - 0$. now f_* is injective, and j_* is injective because S^1 is a retract of $R^2 - 0$, therefore

$g_* = (f \circ g)_* = f_* \circ g_*$ is injective. $\{g = g \circ f\}$

so ~~g_*~~ g is not nullhomotopic

Step III suppose $z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$

with $|a_{n-1}| + \cdots + |a_1| + |a_0| < 1$

Assume it has no root then we can define a map $k: R^2 \to R^2 - 0$ by the equation

$k(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

let h be the restriction of k to S^1, then h is nullhomotopic.

On the other hand, ~~a homoto~~ we shall define a homotopy ~~F~~ H between h & g, but g is not nullhomotopic, we have a contradiction.

We define $H: S^1 \times I \to R^2 - 0$ by the equation

$$H(z,t) = z^n + t(a_{n-1} z^{n-1} + \cdots + a_0)$$

$$= t(z^n + a_{n-1} z^{n-1} + \cdots + a_0) + (1-t) z^n$$

$$= t k(z) + (1-t) g(z)$$

$$\Rightarrow |H(z,t)| = |z^n - \{-t(a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)\}|$$

$$\geq |z|^n - t\{|a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|\}$$

Since $z \in S^1 \Rightarrow |z| = 1$

$$|H(z,t)| = 1 - t(|a_{n-1}| + \cdots + |a_1| + |a_0|) > 0$$

Hence we get a contradiction.

Step IV - Given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Let us choose a real number $c > 0$ and $x = cy$, we obtain

$$(cy)^n + a_{n-1}(cy)^{n-1} + \cdots + a_1(cy) + a_0 = 0$$

$$\Rightarrow \quad y^n + \frac{a_{n-1}}{c} y^{n-1} + \cdots + \frac{a_1}{c^{n-1}} + \frac{a_0}{c^n} = 0$$

If this equation has the root $y = y_0$, then the original equation has the root $x_0 = cy_0$

We choose c large enough that

$$\left|\frac{a_{n-1}}{ac}\right| + \left|\frac{a_{n-2}}{c^2}\right| + \cdots + \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1$$

This completes the proof.

Imp deffinition

Product topology or

Tychnoff Product topology - let $\{(X_i, T_i) : i \in I\}$ be a family of topological space and let X be the cartesian product $\prod_{i \in I} X_i$ and for each $i \in I$, let $\pi_i : X \to X_i$ be the projection function. then the product topology on X is the smallest topology on X which makes each π_i is continuous.

The product topology is sometimes called the tychnoff topology.

(Theorem)

In term of sub-base : Let $\mathcal{T}$ be the product topology on the set $\prod_{i \in I} X_i$, where $\{(X_i, \mathcal{T}_i) : i \in I\}$ is an indexed collection of topological spaces. Then the family of all subsets of the form $\pi_i^{-1}(V_i)$ for $V_i \in \mathcal{T}_i$, $i \in I$ is a sub-base for $\mathcal{T}$. Also the family of all large boxes all of whose sides are open in the respective space is a base for $\mathcal{T}$.

The sub-base and the base given by the above theorem are called respectively the standard sub-base and the standard base for the product topology.

Tychnoff product topology in term of sub-bases :-

Let $\{(X_i, T_i) : i \in I\}$ be arbitrary collection of topological spaces and let $X = \prod_{i \in I} X_i$;

Then the topology T on X which has sub-base the collection $\beta = \{\pi_i^{-1}(V_i) : V_i \in T_i \ \forall i \in I\}$ is called the product topology or Tychnoff topology for X. The space (X, T) is called Tychnoff space.

Projection map :- If $X = \prod\{X_i : i \in I\}$ the function $\pi_i : X \to X_i$ defined by $\pi_i(x) = x_i$ $\forall x \in X$ is said to be projection function.

Tychnoff theorem for compact space :- An arbitrary product of compact spaces is compact in the product topology.

Equivalent Condition for compactness - For a topological space X, the following statement are equivalent.

1. X is compact.
2. Every net in X has a cluster point in X.
3. Every net in X has a convergent subnet in X i.e. a subnet which converges to at least one point in X

F.I.P. for family of closed set :- A family $\mathcal{F}$ of subsets of a set X is said to have the f.i.p. if for any $n \in N$ and $F_1, F_2, \ldots, F_n \in \mathcal{F}$, the intersection $\bigcap_{i=1}^{n} F_i$ is non-empty.

Urysohn "metrization theorem" :- Every regular space X with a countable basis is metrizable.

Distinguish between of the box topologies and Product topology :-

① The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open for X_α for each α and U_α equals X_α except for finitely many values of α.

② The box topology is finer than the product topology.

because - Finite product hold for arbitrary products if we use the product topology, but it is not hold for arbitrary product if we use the box topology.

③ As a result, the product topology is extremely important in mathematics. The box topology is not so important.

Box Topology :- let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$, the collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where U_α is open in X_α for each $\alpha \in J$. The topology generated by this basis is called the box topology.

Condition - First Countable :- Arbitrary product of first countable space is first countable.
(Let $X = \prod_{i \in I} X_i$, then we have to show that X is first countable.)

$\Rightarrow$ Let $f: X \to Y$ be continuous and open and $x_0 \in X$ if X is first countable at x_0, then Y is first countable at $f(x_0)$. Hence first countability is preserved under continuous, open functions.

OR

let $\pi_i : X \to X_i$ be the projection function we know that projection function are continuous and open Also first countability is preserved under continuous function. Hence each X_i is first countable.

Condition for first countable - Let $X = \prod_{i\in I} X_i$ and let $x \in X$. Then X is first countable at x if and only if for each $i \in I$, X_i is first countable at $\pi_i(x)$ and for all except countably many i's, X_i is the only nbd of $\pi_i(x)$ in X_i.
OR (X_i is indiscrete space)

For Second countable - Let $X = \prod_{i\in I} X_i$ ~~& $x \in X$~~. Then X is second countable at x if and only if for each $i \in I$, X_i is second countable at and for all except countably many i's, X_i is indiscrete space.

Projection function are continuous.

Proof - Suppose $X = \prod_{i\in I} X_i$ is endowed with product topology.

Let $\pi_i : X \longrightarrow X_i$ s.t. $\pi_i(x) = x_i$, $\forall x \in X$ be the projection map.

Let G ~~$\in X$~~ be an open set in X_i containing x, then

$\pi_i^{-1}(G) = \{x \in X : \pi_i(x) \in G\}$

$= \{x \in X : x_i \in G\}$

$= \{x \in X_i : x_i \in G\} = G \times X_i$

$=$ an open set in X_i

$\Rightarrow \pi_i^{-1}(G)$

$\Rightarrow$ G is open in X_i $\Rightarrow$ $G \times X_i$ is open in X (X_i is open in X_i)

$\Rightarrow$ $\pi_i^{-1}(G)$ is open in X.

$\Rightarrow$ π_i is continuous.

Smirnov - Metrization Theorem :- A space X is metrizable if and only if it is paracompact and locally finite. metrizable)

Proof - Suppose that X is metrizable, then X is locally metrizable and ~~also~~ it is also paracompact, because every metrizable space is paracompact (by (Stone's theorem)

Conversely - Suppose that X is paracompact and locally metrizable. ~~We have to show that X is metrizable.~~ ~~St~~

~~Since every regular lindelöff sp space~~ is paracompact, so X is regular.

Next proof ~ Same as N.S.M.T. ~ P. 49

We shall show that X has a basis that is countably locally finite. Since X is regular it will then follow from the Nagata-Smirnov theorem that X is metrizable.

N.S.M.T. Conversely part - 49

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